HOMEWORK SET 11
Theory of Condensed Matter
UFV/TKL1/99 lecture by Martin Gmitra
Winter Semester 2023, room: SJSP16

1. [*1 point*] Show that energy of the Cooper pair $\varepsilon = 2\varepsilon_{\rm F} + \dfrac{2\hbar\omega_{\rm D}}{1 - e^{1/Vg(\varepsilon_{\rm F})}}$ is equal for $Vg(\varepsilon_{\rm F}) \ll 1$ to $\varepsilon \simeq 2\varepsilon_{\rm F} - 2\hbar\omega_{\rm D} e^{-1/Vg(\varepsilon_{\rm F})}$ (refer to lecture notes).

2. [*4 points*] Check the following properties for the elements of the unitary matrix diagonalizing the Bogoliubov-de Gennes Hamiltonian (refer to lecture notes)
   a) $|u_k|^2 = \dfrac{1}{2}\left(1 + \dfrac{\varepsilon_k}{E_k}\right)$ and $|v_k|^2 = \dfrac{1}{2}\left(1 - \dfrac{\varepsilon_k}{E_k}\right)$
   b) $|u_k|^2 + |v_k|^2 = 1$
   c) $|u_k|^2 - |v_k|^2 = \dfrac{\varepsilon_k}{E_k}$
   d) $u_k v_k^* = -e^{i(\varphi_\Delta + \pi)} \dfrac{|\Delta|}{2E_k} = \dfrac{\Delta}{2E_k}$

3. [*1 point*] Check the anticommutation for the Bogoliubov quasiparticles $\{b_{\mathbf{k},\sigma}\,,\, b^\dagger_{\mathbf{k}',\sigma'}\} = \delta_{\mathbf{k},\mathbf{k}'}\delta_{\sigma,\sigma'}$.

4. By using the pair operators $\beta^\dagger_{\mathbf{k}} = a^\dagger_{\mathbf{k}\uparrow} a^\dagger_{-\mathbf{k}\downarrow}$ and $\beta_{\mathbf{k}} = a_{-\mathbf{k}\downarrow} a_{\mathbf{k}\uparrow}$
   a) [*1 point*] check that $[\beta_{\mathbf{k}}, \beta^\dagger_{\mathbf{k}'}] = \beta_{\mathbf{k}}\beta^\dagger_{\mathbf{k}'} - \beta^\dagger_{\mathbf{k}'}\beta_{\mathbf{k}} = 0$ for $\mathbf{k} \neq \mathbf{k}'$;
   $[\beta_{\mathbf{k}}, \beta^\dagger_{\mathbf{k}}] = \beta_{\mathbf{k}}\beta^\dagger_{\mathbf{k}} - \beta^\dagger_{\mathbf{k}}\beta_{\mathbf{k}} = 1 - (n_{\mathbf{k}\uparrow} + n_{-\mathbf{k}\downarrow})$, and $[\beta_{\mathbf{k}}, \beta_{\mathbf{k}'}] = [\beta^\dagger_{\mathbf{k}}, \beta^\dagger_{\mathbf{k}'}] = 0$
   b) [*1 point*] when the pair operators satisfy the Bose-Einstein statistics and when they satisfy the Pauli principle?
   c) [2 *points*] show that BCS Hamiltonian $\mathcal{H} = \sum_{\mathbf{k}\sigma} \varepsilon_{\mathbf{k}} a^\dagger_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma} + \sum_{\mathbf{k}\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} a^\dagger_{\mathbf{k}'\uparrow} a^\dagger_{-\mathbf{k}'\downarrow} a_{-\mathbf{k}\downarrow} a_{\mathbf{k}\uparrow}$
   can be written in the form $\mathcal{H}^\beta = \sum_{\mathbf{k}} 2\varepsilon_{\mathbf{k}} \beta^\dagger_{\mathbf{k}}\beta_{\mathbf{k}} + \sum_{\mathbf{k}\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} \beta^\dagger_{\mathbf{k}}\beta_{\mathbf{k}'}$

5. Consider a metal described by the BCS Hamiltonian given in task 4 c) and let $V_{\mathbf{k}\mathbf{k}'} = \lambda|\varepsilon_{\mathbf{k}}||\varepsilon_{\mathbf{k}'}|$ for $|\varepsilon_{\mathbf{k}}| < \hbar\omega_{\rm D}$, $|\varepsilon_{\mathbf{k}'}| < \hbar\omega_{\rm D}$, and $V_{\mathbf{k}\mathbf{k}'} = 0$ otherwise. The $\varepsilon_{\mathbf{k}}$ is measured from the Fermi level.
   a) [*2 extra points*] Solve the gap equation for zero temperature.
   b) [*2 extra points*] What is the criterion for the existence of superconductivity? (gap at $T_{\rm c}$)
   c) [*3 extra points*] Sketch the density of states of excited states as a function of energy.

6. [*5 extra points*] Calculate for lead (Pb) coherence length, value of the specific heat discontinuity ΔC at $T = T_{\rm c}$, and find temperature dependence of the critical field $H_{\rm c}$.
   Hint: look up literature sources at the shared disk.