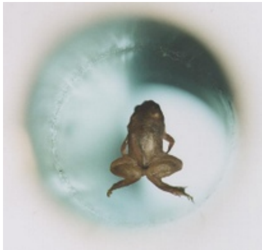
HOMEWORK SET 12
Theory of Condensed Matter
UFV/TKL1/99 lecture by Martin Gmitra
Winter Semester 2023, room: SJSP16

1. [*1 point*] Calculate magnetic moment of an electron (assume $g = 2$). What is the energy difference in the magnetic field of 0.3 Tesla if the spin points parallel or antiparallel to the field? Convert the energy units into a frequency. What is the Larmor precession frequency (*see:* https://en.wikipedia.org/wiki/Larmor_precession) of this electron?

2. [*1 point*] Show that spin precession $\frac{\mathrm{d}\langle\hat{\mathbf{s}}\rangle}{\mathrm{d}t} = \frac{1}{\mathrm{i}\hbar}\langle[\hat{\mathbf{s}}, \mathcal{H}]\rangle = -\frac{g\mu_\mathrm{B}}{\hbar}\langle\hat{\mathbf{s}}\rangle \times \mathbf{B}$ for $\hat{\mathcal{H}} = g\mu_\mathrm{B}\mathbf{B}\cdot\hat{\mathbf{s}}$. Note the similarity to the gyromagnetic ratio in Larmor precessions.

3. Calculate the Pauli paramagnetic, Landau diamagnetic, and the total susceptibility
   a) [*1 point*] of aluminum, using its free electron Fermi wavevector and effective mass (*see tables in Supporting materials*). The experimental value is $\chi_\mathrm{exp} = 2.2 \times 10^{-5}$.
   b) [*1 point*] compare the quantities to bismuth.



4. [*2 points*] Estimate amount of water in the body of a levitating frog of 22 grams of its weight. Assume a thin solenoid of length $L = 10\,\mathrm{cm}$ for which $B\nabla B \approx B_c^2/L$, and field in solenoid center $B_c = 13.6\,\mathrm{T}$.

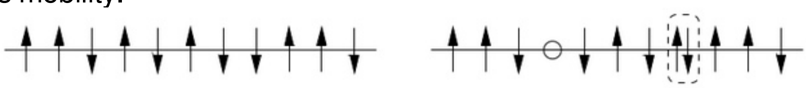
5. Show that the operator $\hat{s}_{\theta,\phi} = \sin\theta\cos\phi\,\hat{s}_x + \sin\theta\sin\phi\,\hat{s}_y + \cos\theta\,\hat{s}_z$, representing the spin operator for the spin components along a direction determined by the spherical and polar angles,
   a) [*1 point*] has eigenvalues $\pm\hbar/2$ and eigenstates of the form $|\uparrow\rangle = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)e^{\mathrm{i}\phi} \end{pmatrix}$ and $|\downarrow\rangle = \begin{pmatrix} \sin(\theta/2) \\ -\cos(\theta/2)e^{\mathrm{i}\phi} \end{pmatrix}$
   b) [*1 point*] show that $\hat{s}^2_{\theta,\phi} = \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

6. [*2 points*] An electron in a magnetic field along the *z*-direction has the Hamiltonian $\hat{\mathcal{H}} = g\mu_\mathrm{B}\mathbf{B}\cdot\hat{\mathbf{s}} = g\mu_\mathrm{B}B\hat{s}_z$. The spin angular momentum is defined as $\hat{\mathbf{s}} = \hbar\hat{\boldsymbol{\sigma}}/2$. The solution of the time-dependent Schrödinger equation $\hat{\mathcal{H}}\psi(t) = \mathrm{i}\hbar\,\mathrm{d}\psi(t)/\mathrm{d}t$ has the form $\psi(t) = \exp(-\mathrm{i}\hat{\mathcal{H}}t/\hbar)\psi(0)$. Using the fact that $(\boldsymbol{\sigma}\cdot\mathbf{a})^2 = |\mathbf{a}|^2$, where $\mathbf{a} = (a_1, a_2, a_3)^\mathrm{T}$ is a three-component vector and $\boldsymbol{\sigma} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$ is the vector of Pauli matrices, show that $\exp(\mathrm{i}\alpha\sigma_j) = \mathbb{I}\cos\alpha + \mathrm{i}\sigma_j\sin\alpha$, where $\mathbb{I}$ is the identity matrix, σ_j is the one of Pauli matrices, and α is a real number. Show that if $\psi(t)$ is written as a spinor

$$\psi(t) = \begin{pmatrix} \exp(-\mathrm{i}g\mu_\mathrm{B}Bt/2\hbar) & 0 \\ 0 & \exp(\mathrm{i}g\mu_\mathrm{B}Bt/2\hbar) \end{pmatrix}\psi(0)$$

   and using results from 5a) the evolution of the spin state is such that the expected value of θ is conserved but ϕ is rotates with the angular frequency $ge\mu_\mathrm{B}/2m$.

7. [*5 extra points*] Consider Hubbard model for a single orbital per site and nearest neighbor hopping in insulating limit $t \ll U$. One of the lowest excited states is created by one empty and one doubly occupied site. The empty "holon" and doubly occupied site "doublon" become mobile. Calculate degeneracy of the excited state if the system has N sites and estimate energy of its mobility.



ground state energy sector

excited state energy sector

HOMEWORK SET 12
Theory of Condensed Matter
UFV/TKL1/99 lecture by Martin Gmitra
Winter Semester 2023, room: SJSP16

8. [*5 extra points*] Find the Curie constant C_{Curie} in the so called *Curie law* which states that the susceptibility is inversely proportional to temperature, $\chi = C_{\text{Curie}}/T$. For derivation consider the number of electrons per unit volume of each spin state is given by

$$n_\sigma = \frac{1}{2}\int_0^\infty g(E + \sigma\mu_{\text{B}}B)f(E)\mathrm{d}E$$

where $\sigma = 1$ for spin up, $\sigma = -1$ for spin down, $g(E)$ is the corresponding density of states, and for the distribution function consider non-degenerate limit $f(E) \approx \exp(-(E-\mu)/k_{\text{B}}T)$. The magnetization is given by $M = \mu_{\text{B}}(n_+ - n_-)$. What is the susceptibility equal to in the degenerate limit at $T = 0$?