Cauchy principal value

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1. Idea

The *Cauchy principal value* of a function which is integrable on the complement of one point is, if it exists, the limit of the integrals of the function over subsets in the complement of this point as these integration domains tend to that point *symmetrically* from all sides.

One also subsumes the case that the "point" is "at infinity", hence that the function is integrable over every bounded domain. In this case the Cauchy principal value is the limit, if it exists, of the integrals of the function over bounded domains, as their bounds tend *symmetrically* to infinity.

The operation of sending a compactly supported smooth function (bump function) to Cauchy principal value of its pointwise product with a function f that may be singular at the origin defines a distribution, usually denoted $\mathrm{PV}(f)$.

When the Cauchy principal value exists but the full integral does not (hence when the full integral "diverges") one may think of the Cauchy principal value as "exracting a finite value from a diverging quantity". This is similar to the *intuition* of the early days of renormalization in perturbative quantum field theory (Schwinger-Tomonaga-Feynman-Dyson), but one has to be careful not to carry this analogy too far.

One point where the Cauchy principal value really does play a key role in perturbative quantum field theory is in the computation of Green functions (propagators) for the Klein-Gordon operator and the Dirac operator. See remark 3.8 below and see at *Feynman propagator* for more on this.

2. Definition

As an integral

***Definition 2.1**. (Cauchy principal value of an integral over the real line)*

Let $f : \mathbb{R} \to \mathbb{R}$ be a function on the real line such that for every positive real number ϵ its restriction to $\mathbb{R} \setminus (-\epsilon, \epsilon)$ is integrable. Then the *Cauchy principal value* of f is, if it exists, the limit

$$\mathrm{PV}(f) \coloneqq \lim_{\epsilon \to 0} \int_{\mathbb{R} \setminus (-\epsilon,\epsilon)} f(x)\, dx \,.$$

As a distribution

Definition 2.2. (Cauchy principal value as distribution on the real line)

Let $f : \mathbb{R} \to \mathbb{R}$ be a function on the real line such that for all bump functions $b \in C^\infty_{cp}(\mathbb{R})$ the Cauchy principal value of the pointwise product function $f b$ exists, in the sense of def. 2.1. Then this assignment

$$\mathrm{PV}(f) \,:\, b \mapsto \mathrm{PV}(f b)$$

defines a distribution $\mathrm{PV}(f) \in \mathcal{D}'(\mathbb{R})$.

3. Examples

The principal value of $1/x$

Example 3.1. Let $f : \mathbb{R} \to \mathbb{R}$ be an integrable function which is symmetric, in that $f(-x) = f(x)$ for all $x \in \mathbb{R}$. Then the principal value integral (def. 2.1) of $x \mapsto \frac{f(x)}{x}$ exists and is zero:

$$\lim_{\epsilon \to 0} \int_{\mathbb{R} \setminus (-\epsilon,\epsilon)} \frac{f(x)}{x}\, dx \;=\; 0$$

This is because, by the symmetry of f and the skew-symmetry of $x \mapsto 1/x$, the the two contributions to the integral are equal up to a sign:

$$\int_{-\infty}^{-\epsilon} \frac{f(x)}{x}\, dx \;=\; - \int_{\epsilon}^{\infty} \frac{f(x)}{x}\, dx \,.$$

Example 3.2. The principal value distribution $\mathrm{PV}\left(\frac{1}{x}\right)$ (def. 2.2) solves the distributional equation

$$x \,\mathrm{PV}\left(\frac{1}{x}\right) = 1 \qquad \in \mathcal{D}'(\mathbb{R}^1) \,. \tag{1}$$

Since the delta distribution $\delta \in \mathcal{D}'(\mathbb{R}^1)$ solves the equation

$$x\delta(x) = 0 \qquad \in \mathcal{D}'(\mathbb{r}^1)$$

we have that more generally every linear combination of the form

$$F(x) \coloneqq \mathrm{PV}(1/x) + c\delta(x) \qquad \in \mathcal{D}'(\mathbb{R}^1) \tag{2}$$

for $c \in \mathbb{C}$, is a distributional solution to $xF(x) = 1$.

The wave front set of all these solutions is

$$\mathrm{WF}(\mathrm{PV}(1/x) + c\delta(x)) \;=\; \{(0,k) \mid k \in \mathbb{R}^\ast \setminus \{0\}\} \,.$$

Proof. The first statement is immediate from the definition: For $b \in C^\infty_c(\mathbb{R}^1)$ any bump function we have that

$$\left\langle x\, \mathrm{PV}\left(\frac{1}{x}\right), b\right\rangle := \lim_{\epsilon \to 0} \int_{\mathbb{R}^1 \setminus (-\epsilon,\epsilon)} \frac{x}{x} b(x)\, dx$$

$$= \int b(x) dx$$

$$= \langle 1, b\rangle$$

Regarding the second statement: It is clear that the wave front set is concentrated at the origin. By symmetry of the distribution around the origin, it must contain both directions. ▮

***Proposition 3.3**. In fact (2) is the most general distributional solution to (1).*

This follows by the characterization of extension of distributions to a point, see there at this prop. (Hörmander 90, thm. 3.2.4)

Definition 3.4**. **(integration against inverse variable with imaginary offset)

Write

$$\frac{1}{x + i0^{\pm}} \;\in\; \mathcal{D}'(\mathbb{R})$$

for the distribution which is the limit in $\mathcal{D}'(\mathbb{R})$ of the non-singular distributions which are given by the smooth functions $x \mapsto \frac{1}{x \pm i\epsilon}$ as the positive real number ϵ tends to zero:

$$\frac{1}{x + i0^{\pm}} \;:=\; \lim_{\substack{\epsilon \in (0,\infty) \\ \epsilon \to 0}} \frac{1}{x \pm i\epsilon}$$

hence the distribution which sends $b \in C^\infty(\mathbb{R}^1)$ to

$$b \mapsto \int_{\mathbb{R}} \frac{b(x)}{x \pm i\epsilon}\, dx \;.$$

Proposition 3.5**. **(Cauchy principal value equals integration with imaginary offset plus delta distribution)

The Cauchy principal value distribution $\mathrm{PV}\left(\frac{1}{x}\right) \in \mathcal{D}'(\mathbb{R})$ (def. 2.2) is equal to the sum of the integration over $1/x$ with imaginary offset (def. 3.4) and a delta distribution.

$$\mathrm{PV}\left(\frac{1}{x}\right) \;=\; \frac{1}{x + i0^{\pm}} \pm i\pi\delta \;.$$

In particular, by prop. 3.2 this means that $\frac{1}{x+i0^{\pm}}$ solves the distributional equation

$$x \frac{1}{x + i0^{\pm}} \;=\; 1 \qquad \in \mathcal{D}'(\mathbb{R}^1) \;.$$

***Proof**.* Using that

$$\frac{1}{x \pm i\epsilon} = \frac{x \mp i\epsilon}{(x + i\epsilon)(x - i\epsilon)}$$

$$= \frac{x \mp i\epsilon}{(x^2 + \epsilon^2)}$$

we have for every bump function $b \in C^\infty_{\mathrm{cp}}(\mathbb{R}^1)$

$$\lim_{\epsilon\to 0} \int_{\mathbb{R}^1} \frac{b(x)}{x \pm i\epsilon}\, dx = \underbrace{\lim_{\epsilon\to 0} \int_{\mathbb{R}^1} \frac{x^2}{x^2+\epsilon^2} \frac{b(x)}{x}\, dx}_{(A)} \mp i\pi \underbrace{\lim_{\epsilon\to 0} \int_{\mathbb{R}^1} \frac{1}{\pi} \frac{\epsilon}{x^2+\epsilon^2}\, b(x)\, dx}_{(B)}$$

Since

$$\begin{array}{ccc} & \frac{x^2}{x^2+\epsilon^2} & \\ {}^{|x|<\epsilon}_{\epsilon\to 0} \swarrow & & \searrow {}^{|x|>\epsilon}_{\epsilon\to 0} \\ 0 & & 1 \end{array}$$

it is plausible that $(A) = \mathrm{PV}\left(\frac{b(x)}{x}\right)$, and similarly that $(B) = b(0)$. In detail:

$$\begin{aligned} (A) &= \lim_{\epsilon\to 0} \int_{\mathbb{R}^1} \frac{x}{x^2+\epsilon^2}\, b(x) dx \\ &= \lim_{\epsilon\to 0} \int_{\mathbb{R}^1} \frac{d}{dx}\left(\tfrac{1}{2}\ln(x^2+\epsilon^2)\right) b(x) dx \\ &= -\tfrac{1}{2}\lim_{\epsilon\to 0} \int_{\mathbb{R}^1} \ln(x^2+\epsilon^2) \frac{db}{dx}(x) dx \\ &= -\tfrac{1}{2} \int_{\mathbb{R}^1} \ln(x^2) \frac{db}{dx}(x) dx \\ &= -\int_{\mathbb{R}^1} \ln(|x|) \frac{db}{dx}(x) dx \\ &= -\lim_{\epsilon\to 0} \int_{\mathbb{R}^1 \setminus (-\epsilon,\epsilon)} \ln(|x|) \frac{db}{dx}(x) dx \\ &= \lim_{\epsilon\to 0} \int_{\mathbb{R}^1 \setminus (-\epsilon,\epsilon)} \frac{1}{x}\, b(x) dx \\ &= \mathrm{PV}\left(\frac{b(x)}{x}\right) \end{aligned}$$

and

$$\begin{aligned} (B) &= \tfrac{1}{\pi}\lim_{\epsilon\to 0} \int_{\mathbb{R}^1} \frac{\epsilon}{x^2+\epsilon^2}\, b(x)\, dx \\ &= \tfrac{1}{\pi}\lim_{\epsilon\to 0} \int_{\mathbb{R}^1} \left(\frac{d}{dx}\arctan\left(\frac{x}{\epsilon}\right)\right) b(x)\, dx \\ &= -\tfrac{1}{\pi}\lim_{\epsilon\to 0} \int_{\mathbb{R}^1} \arctan\left(\frac{x}{\epsilon}\right) \frac{db}{dx}(x)\, dx \\ &= -\frac{1}{2} \int_{\mathbb{R}^1} \mathrm{sgn}(x) \frac{db}{dx}(x)\, dx \\ &= b(0) \end{aligned}$$

where we used that the derivative of the arctan function is $\frac{d}{dx}\arctan(x) = 1/(1+x^2)$ and that

$\lim\limits_{\epsilon \to +\infty} \arctan(x/\epsilon) = \frac{\pi}{2}\mathrm{sgn}(x)$ is proportional to the sign function. ▮

Example. ***(Fourier integral formula for step function)***

The Heaviside distribution $\Theta \in \mathcal{D}'(\mathbb{R})$ is equivalently the following Cauchy principal value:

$$\begin{aligned}\Theta(x) &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{i\omega x}}{\omega - i0^+} \\ &:= \lim_{\epsilon \to 0^+} \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{i\omega x}}{\omega - i\epsilon}\, d\omega\,,\end{aligned}$$

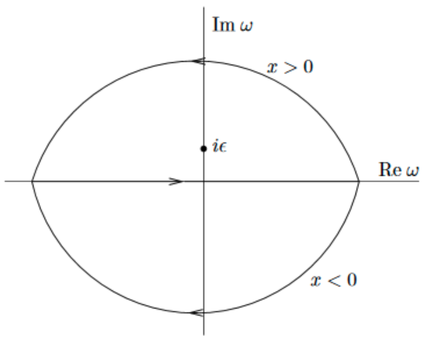
where the limit is taken over sequences of positive real numbers $\epsilon \in (-\infty, 0)$ tending to zero.

Proof. We may think of the integrand $\frac{e^{i\omega x}}{\omega - i\epsilon}$ uniquely extended to a holomorphic function on the complex plane and consider computing the given real line integral for fixed ϵ as a contour integral in the complex plane.

If $x \in (0, \infty)$ is positive, then the exponent

$$i\omega x = -\mathrm{Im}(\omega)x + i\,\mathrm{Re}(\omega)x$$

has negative real part for *positive* imaginary part of ω. This means that the line integral equals the complex contour integral over a contour $C_+ \subset \mathbb{C}$ closing in the upper half plane. Since $i\epsilon$ has positive imaginary part by construction, this contour does encircle the pole of the integrand $\frac{e^{i\omega x}}{\omega - i\epsilon}$ at $\omega = i\epsilon$. Hence by the Cauchy integral formula in the case $x > 0$ one gets



$$\begin{aligned}\lim_{\epsilon \to 0^+} \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{i\omega x}}{\omega - i\epsilon}\, d\omega &= \lim_{\epsilon \to 0^+} \frac{1}{2\pi i}\oint_{C_+} \frac{e^{i\omega x}}{\omega - i\epsilon}\, d\omega \\ &= \lim_{\epsilon \to 0^+} \left(e^{i\omega x}\vert_{\omega = i\epsilon}\right) \\ &= \lim_{\epsilon \to 0^+} e^{-\epsilon x} \\ &= e^0 = 1\end{aligned}.$$

Conversely, for $x < 0$ the real part of the integrand decays as the *negative* imaginary part increases, and hence in this case the given line integral equals the contour integral for a contour $C_- \subset \mathbb{C}$ closing in the lower half plane. Since the integrand has no pole in the lower half plane, in this case the Cauchy integral formula says that this integral is zero. ▮

Conversely, by the Fourier inversion theorem, the Fourier transform of the Heaviside distribution is the Cauchy principal value as in prop. 3.5:

Example 3.6. ***(relation to Fourier transform of Heaviside distribution / Schwinger parameterization)***

$$\begin{aligned}\widehat{\Theta}(x) &= \int_0^{\infty} e^{ikx}\,\mathrm{dk} \\ &= i\,\frac{1}{x + i0^+}\end{aligned}$$

Here the second equality is also known as *complex Schwinger parameterization*.

Proof. As generalized functions consider the limit with a decaying component:

$$
\begin{aligned}
\int_0^\infty e^{ikx}\,\mathrm{dk} &= \lim_{\epsilon\to 0^+}\int_0^\infty e^{ikx-\epsilon k}\,\mathrm{dk} \\
&= -\lim_{\epsilon\to 0^+}\frac{1}{ix-\epsilon} \\
&= i\frac{1}{x+i0^+}
\end{aligned}
$$

▮

The principal value of $1/(q(x)+m^2)$

Let $q:\mathbb{R}^n \to \mathbb{R}$ be a non-degenerate real quadratic form analytically continued to a real quadratic form

$$
q : \mathbb{C}^n \longrightarrow \mathbb{C}\,.
$$

Write Δ for the determinant of q

Write q^* for the induced quadratic form on dual vector space. Notice that q (and hence a^*) are assumed non-degenerate but need not necessarily be positive or negative definite.

***Proposition 3.7*. (*Fourier transform* of principal value of power of *quadratic form*)**

Let $m\in\mathbb{R}$ be any real number, and $\kappa\in\mathbb{C}$ any complex number. Then the Fourier transform of distributions of $1/(q+m^2+i0^+)^\kappa$ is

$$
\widehat{\left(\frac{1}{q+m^2+i0^+}\right)} \;=\; \frac{2^{1-\kappa}(\sqrt{2\pi})^n m^{n/2-\kappa}}{\Gamma(\kappa)\sqrt{\Delta}}\,\frac{K_{n/2-\kappa}\left(m\sqrt{q^*-i0^+}\right)}{\left(\sqrt{q^*-i0^+}\right)^{n/2-\kappa}}\,,
$$

where

1. *Γ deotes the Gamma function*
2. *K_ν denotes the modified Bessel function.*

Notice that $K_\nu(a)$ diverges for $a\to 0$ as $a^{-\nu}$ (DLMF 10.30.2).

(Gel'fand-Shilov 66, III 2.8 (8) and (9), p 289)

***Example 3.8*. (*Feynman propagator*)**

Let $q \coloneqq \eta^{-1}$ be the dual Minkowski metric in dimension $p+1$. Then

$$
\Delta_F(x) \;\propto\; \widehat{\frac{1}{-\eta^{-1}(k,k)-m^2+i0^+}}
$$

is the Feynman propagator for the Klein-Gordon equation on Minkowski spacetime. In this case prop. 3.7 implies that its singular support is the light cone $\{x\in\mathbb{R}^{p,1}\mid \eta(x,x)=0\}$.

The Fourier transform of $\delta(q+m^2)$

Let $q:\mathbb{R}^n \to \mathbb{R}$ be a non-degenerate real quadratic form analytically continued to a real quadratic form

$$
q : \mathbb{C}^n \longrightarrow \mathbb{C}\,.
$$

Write Δ for the determinant of q. Write $t\in\mathbb{N}$ for the number of negative eigenvalues.

Write q^* for the induced quadratic form on dual vector space. Notice that q (and hence a^*) are assumed non-degenerate but need not necessarily be positive or negative definite.

***Proposition 3.9*. (Fourier transform of delta distribution applied to mass shell)**

Let $m \in \mathbb{R}$, then the Fourier transform of distributions of the delta distribution δ applied to the "mass shell" $q + m^2$ is

$$\widehat{\delta(q+m^2)} \;=\; -\frac{i}{\sqrt{|\Delta|}} \left(e^{i\pi t/2} \frac{K_{n/2-1}\left(m\sqrt{q^*+i0^+}\right)}{\left(\sqrt{q^*+i0^+}\right)^{n/2-1}} \;-\; e^{-i\pi t/2} \frac{K_{n/2-1}\left(m\sqrt{q^*-i0^+}\right)}{\left(\sqrt{q^*-i0^+}\right)^{n/2-1}} \right),$$

where K_ν denotes the modified Bessel function of order ν.

Notice that $K_\nu(a)$ diverges for $a \to 0$ as $a^{-\nu}$ (DLMF 10.30.2).

(Gel'fand-Shilov 66, III 2.11 (7), p 294)

***Example 3.10*. (causal propagator)**

Let $q \coloneqq \eta^{-1}$ be the dual Minkowski metric in dimension $p+1$. Then

$$\Delta_S(x) \;\propto\; \widehat{\delta(-\eta(k,k) - m^2)\,\mathrm{sgn}(k_0)}$$

is the causal propagator for the Klein-Gordon equation on Minkowski spacetime. In this case prop. 3.9 implies that its singular support is the light cone $\{x \in \mathbb{R}^{p,1} \mid \eta(x,x) = 0\}$.

4. Related concepts

- zeta function regularization

5. References

Named after Augustin Cauchy

- Ram Kanwal, section 8.3 of *Linear Integral Equations* Birkhäuser 1997

Detailed discussion of relation to Bessel functions is in

- I. M. Gel'fand, G. E. Shilov, *Generalized functions*, 1–5 , Acad. Press (1966–1968) transl. from И. М. Гельфанд, Г. Е. Шилов Обобщенные функции, вып. 1-3, М.:Физматгиз, 1958; 1: Обобщенные функции и действия над ними, 2: Пространства основных обобщенных функций, 3: Некоторые вопросы теории дифференциальных уравнений

References on homogeneous distributions

- Lars Hörmander, *The Analysis of Linear Partial Differential Operators I* (Springer, 1990, 2nd ed.)

See also

- Wikipedia, *Cauchy principal value*
- Wikipedia, *Hadamard principal value*