9.2 The Drude model for metals

For simple metals, we can apply the theory of Paul Drude, which was developed around 1900.[1] This model is that of a free electron gas with "dirt" used to scatter electrons, resulting in a scattering time of τ. To analyze this system, we use the relaxation time approximation. From this model we can derive the dielectric function by starting with the equation of motion for the electron including the relaxation time τ. Hence,

$$m\ddot{\mathbf{x}} = -e\mathbf{E} - \frac{m\dot{\mathbf{x}}}{\tau},$$

or (by taking the time Fourier transform)

$$-m\omega^2\mathbf{x}(\omega) = -im\omega\gamma\,\mathbf{x}(\omega) - e\mathbf{E}(\omega), \tag{9.25}$$

where $\gamma \equiv \frac{1}{\tau}$. Solving for $\mathbf{x}$ gives us

$$m\mathbf{x}(\omega) = \frac{-e\mathbf{E}(\omega)}{-\omega^2 + i\omega\gamma}. \tag{9.26}$$

The polarization per unit volume $\mathbf{P}(\omega)$ is then given by

$$\mathbf{P}(\omega) = -ne\mathbf{x}(\omega) = \alpha\mathbf{E}(\omega). \tag{9.27}$$

We may obtain $\epsilon(\omega)$ through the relation

$$\epsilon = 1 + 4\pi\alpha. \tag{9.28}$$

This allows us to identify the real and imaginary parts of ϵ for this model as

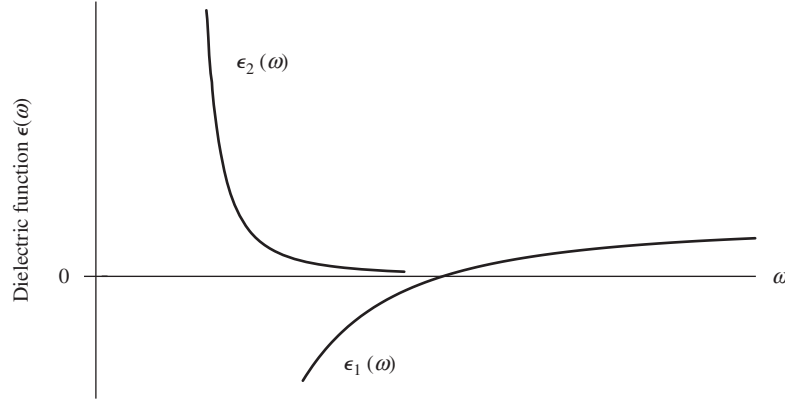$$\epsilon_1(\omega) = 1 - \frac{\omega_p^2\tau^2}{1 + \omega^2\tau^2} \tag{9.29}$$

and

$$\epsilon_2(\omega) = \frac{\omega_p^2\tau}{\omega(1 + \omega^2\tau^2)}, \tag{9.30}$$

where ω_p is the plasma frequency $\sqrt{\frac{4\pi ne^2}{m}}$. These functions are shown in Fig. 9.2.
This calculation also yields

$$\sigma_1(\omega) = \frac{\omega\epsilon_2(\omega)}{4\pi} = \frac{\omega_p^2\tau}{4\pi(1 + \omega^2\tau^2)}. \tag{9.31}$$

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[1] P. Drude, "Zur Elektronentheorie der metalle," *Ann. Physik* 306(1900), 566.

Figure 9.2 Schematic figure for $\epsilon_1(\omega)$ and $\epsilon_2(\omega)$ within the Drude theory. At high ω, $\epsilon_1(\omega) \to 1$.

To get all the desired optical properties, we need only use the relations for the optical constants n and k. Thus, with two material parameters, γ (typical value $\sim 10^{13}$/sec) and ω_p (typical value $\sim 10^{15}$/sec), we can calculate the optical properties of simple metals.

We now explore the reflectivity of a metal in three characteristic frequency regions. For region I, which is for very low ω, we consider $\omega \ll \omega_p$ and $\omega\tau \ll 1$, corresponding to many scatterings in one EM wave period. In this regime, for the typical values given above, the equations become

$$\epsilon_1(\omega) \approx -\omega_p^2 \tau^2 \sim -10^4 \tag{9.32}$$

and

$$\epsilon_2(\omega) \approx \frac{\omega_p^2 \tau}{\omega} = \frac{\omega_p^2 \tau^2}{\omega\tau} \gg |\epsilon_1(\omega)|. \tag{9.33}$$

This dominance of $\epsilon_2(\omega)$ allows us to calculate $N(\omega)$ as follows

$$N(\omega) = \frac{\omega_p \tau}{\sqrt{\omega\tau}} \left(\frac{1+i}{\sqrt{2}} \right). \tag{9.34}$$

This condition sets the real and imaginary parts of $N(\omega)$ equal and, since $N = n + ik$, then $|n| = |k| \gg 1$. Thus, we can calculate the reflection or reflectivity as

$$R = \frac{(n-1)^2 + k^2}{(n+1)^2 + k^2} \approx 1 - \frac{2}{n} = 1 - \frac{2\sqrt{2\omega\tau}}{\omega_p \tau} \approx 1, \tag{9.35}$$

indicating a large reflectivity in this regime. The reflectivity can be written in terms of the conductivity, using Eq. (9.14), as

$$R(\omega) = 1 - \sqrt{\frac{2\omega}{\pi \sigma_1(0)}}, \tag{9.36}$$

which is the reason why highly conducting metals ($\sigma \gg 1$) reflect well at low ω. Equation (9.36) is often written as

$$1 - R(\omega) \sim \sqrt{\omega}, \tag{9.37}$$

and is known as the Hagen–Rubens relation,[2] and this low-frequency region is referred to as the Hagen–Rubens region.

For region II, $\omega\tau \gg 1$ and $\omega < \omega_p$, corresponding to the electron oscillating in the EM field many times before being scattered. The corresponding equations are

$$\epsilon_1(\omega) = 1 - \frac{\omega_p^2 \tau^2}{1 + \omega^2 \tau^2} \approx 1 - \frac{\omega_p^2}{\omega^2} \tag{9.38}$$

and

$$\epsilon_2(\omega) \approx \frac{\omega_p^2}{\omega^2(\omega\tau)} \ll |\epsilon_1(\omega)|. \tag{9.39}$$

This gives a reflectivity of

$$R(\omega) \approx 1 - \frac{2}{\omega_p \tau} \approx 1, \tag{9.40}$$

and explains why simple metals reflect in this frequency range as well. The reflectance is independent of ω, which explains the shininess and colorlessness of many metals.

For region III, which comprises very high $\omega, \omega > \omega_p$, the corresponding equations are

$$\epsilon_1(\omega) = 1 - \frac{\omega_p^2 \tau^2}{1 + \omega^2 \tau^2} > 0, \tag{9.41}$$

with

$$k(\omega) \to 0 \tag{9.42}$$

and

$$n(\omega) = \sqrt{\epsilon_1} \to 1, \tag{9.43}$$

which implies that the metal is transparent. Figure 9.3 summarizes these results with a plot of $1 - R(\omega)$ vs. ω. The abrupt increase in the transmission (since $k(\omega) = 0$ in this high ω range) near ω_p, as shown in Fig. 9.3, is an excellent experimental measure of the plasma frequency.

For the Hagen–Rubens region, we can also estimate other relevant properties. The absorption coefficient $\eta = \frac{2\omega k}{c}$ becomes very large because

$$k(\omega) = \frac{\omega_p \tau}{\sqrt{2\omega\tau}}. \tag{9.44}$$

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[2] The relation was discovered by Ernst B. Hagen and Heinrich Rubens in 1903.
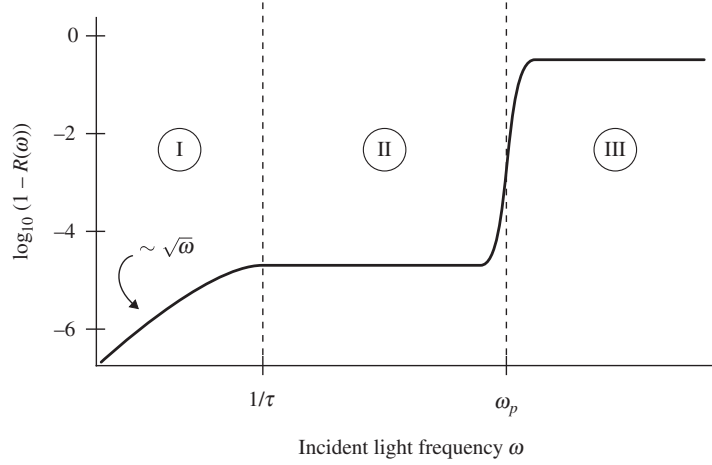
Figure 9.3 Frequency dependence of $\log_{10}(1 - R(\omega))$ within the Drude theory.

Also, the EM wave skin depth

$$\delta = \frac{c}{\omega k} = \frac{c}{\omega}\frac{\sqrt{2\omega\tau}}{\omega_p\tau} \tag{9.45}$$

can be related to the dc conductivity $\sigma_1(0) = \frac{ne^2\tau}{m}$, and we have

$$\delta = \frac{c}{\sqrt{2\pi\,\sigma_1(0)\omega}} \sim \omega^{-\frac{1}{2}}. \tag{9.46}$$

Hence, the simple Drude model, which is based on intraband transitions arising from scattering centers, gives the essential features of the optical properties of simple metals.

9.3 The transverse dielectric function

As discussed in Section 9.1, when examining the physics of optical properties, the appropriate dielectric function to use for determining response functions for transverse EM probes is the "transverse dielectric function." In our previous discussions, in Chapter 8 for example, we considered a longitudinal field with $\mathbf{q} \parallel \mathbf{E}$, because we assumed that

$$\nabla^2\phi = -4\pi\,\delta\rho \implies \nabla \cdot \mathbf{E} = -4\pi\,\delta\rho. \tag{9.47}$$

As $q \to 0$, this approximation is acceptable for transverse EM waves in a cubic or homogeneous system. However, more generally for an optical response, we need to consider a current–current response function derivation of the self-consistent field dielectric function.