Appendix A

SI Units and Gaussian Units

A.1 Conversion of Amounts

All factors of 3 (apart from exponents) should, for accurate work, be replaced by 2.99792456, arising from the numerical value of the velocity of light. [43]

Physical quantity	Symbol	SI(MKSA)		Gaussian
Length	l	1 meter (m)	10^2	centimeters (cm)
Mass	m	1 kilogram (kg)	10^3	grams (gm or g)
Time	t	1 second (sec or s)	1	second (sec or s)
Frequency	f	1 hertz (Hz)	1	hertz (Hz)
Force	F	1 newton (N)	10^5	dynes
Work, Energy	W, U	1 joule (J)	10^7	ergs
Power	P	1 watt (W)	10^7	ergs/s
Charge	q	1 coulomb (C)	3×10^9	statcoulombs
Charge density	ϱ	1 C/m^3	3×10^3	statcoul/cm^3
Current	I	1 ampere (A)	3×10^9	statamperes
Current density	J	1 A/m^2	3×10^5	statamp/cm^2
Potential	φ	1 volt (V)	$10^{-2}/3$	statvolt
Electric field	E	1 V/m	$10^{-4}/3$	statvolt/cm
Electric induction	D	1 C/m^2	$12\pi \times 10^5$	statvolt/cm
Polarization	P	1 C/m^2	3×10^5	moment/cm^3
Magnetic flux	Φ	1 weber (Wb)	10^8	maxwell (Mx)
Magnetic induction	B	1 tesla (T)	10^4	gauss (Gs)
Magnetic field	H	1 A/m	$4\pi \times 10^{-3}$	oersted (Oe)
Magnetization	M	1 A/m	10^{-3}	moment/cm^3
Conductance	G	1 siemens (S)	9×10^{11}	cm/s
Conductivity	σ	1 S/m	9×10^9	1/s
Resistance	R	1 ohm (Ω)	$10^{-11}/9$	s/cm
Capacitance	C	1 farad (F)	9×10^{11}	cm
Inductance	L	1 henry (H)	10^9	cm

A.2 Formulas in SI (MKSA) Units and Gaussian Units

Name of formula	SI (MKSA)	Gaussian
Maxwell equations	$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$ $\nabla \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{J}$ $\nabla \cdot \boldsymbol{D} = \varrho$ $\nabla \cdot \boldsymbol{B} = 0$	$\nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}$ $\nabla \times \boldsymbol{H} = \frac{1}{c}\frac{\partial \boldsymbol{D}}{\partial t} + \frac{4\pi}{c}\boldsymbol{J}$ $\nabla \cdot \boldsymbol{D} = 4\pi\varrho$ $\nabla \cdot \boldsymbol{B} = 0$
Lorentz force	$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$	$\boldsymbol{F} = q(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v} \times \boldsymbol{B})$
Constitutional equations	$\boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P} = \epsilon \boldsymbol{E}$ $\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M}) = \mu \boldsymbol{H}$ $\boldsymbol{J} = \gamma \boldsymbol{E}$	$\boldsymbol{D} = \boldsymbol{E} + 4\pi \boldsymbol{P} = \epsilon \boldsymbol{E}$ $\boldsymbol{B} = \boldsymbol{H} + 4\pi \boldsymbol{M} = \mu \boldsymbol{H}$ $\boldsymbol{J} = \gamma \boldsymbol{E}$
Constitutional parameters	$\epsilon = \epsilon_0(1 + \chi_e) = \epsilon_0 \epsilon_r$ $\mu = \mu_0(1 + \chi_m) = \mu_0 \mu_r$	$\epsilon = 1 + 4\pi\chi_e = \epsilon_r$ $\mu = 1 + 4\pi\chi_m = \mu_r$
Boundary equations	$\boldsymbol{n} \times (\boldsymbol{E}_2 - \boldsymbol{E}_1) = 0$ $\boldsymbol{n} \times (\boldsymbol{H}_2 - \boldsymbol{H}_1) = \boldsymbol{J}_s$ $\boldsymbol{n} \cdot (\boldsymbol{D}_2 - \boldsymbol{D}_1) = \varrho_s$ $\boldsymbol{n} \cdot (\boldsymbol{B}_2 - \boldsymbol{B}_1) = 0$	$\boldsymbol{n} \times (\boldsymbol{E}_2 - \boldsymbol{E}_1) = 0$ $\boldsymbol{n} \times (\boldsymbol{H}_2 - \boldsymbol{H}_1) = \frac{4\pi}{c}\boldsymbol{J}_s$ $\boldsymbol{n} \cdot (\boldsymbol{D}_2 - \boldsymbol{D}_1) = 4\pi\varrho_s$ $\boldsymbol{n} \cdot (\boldsymbol{B}_2 - \boldsymbol{B}_1) = 0$
Coulomb's law	$\boldsymbol{E} = \frac{1}{4\pi\epsilon}\int_V \frac{\varrho}{r^2}\hat{\boldsymbol{r}}dV'$ $\varphi = \frac{1}{4\pi\epsilon}\int_V \frac{\varrho}{r}dV'$	$\boldsymbol{E} = \frac{1}{\epsilon}\int_V \frac{\varrho}{r^2}\hat{\boldsymbol{r}}dV'$ $\varphi = \frac{1}{\epsilon}\int_V \frac{\varrho}{r}dV'$
Biot-Savart law	$\boldsymbol{B} = \frac{\mu}{4\pi}\int_V \frac{\boldsymbol{J} \times \hat{\boldsymbol{r}}}{r^2}dV'$ $\boldsymbol{A} = \frac{\mu}{4\pi}\int_V \frac{\boldsymbol{J}}{r}dV'$	$\boldsymbol{B} = \frac{\mu}{c}\int_V \frac{\boldsymbol{J} \times \hat{\boldsymbol{r}}}{r^2}dV'$ $\boldsymbol{A} = \frac{\mu}{c}\int_V \frac{\boldsymbol{J}}{r}dV'$
Poison equations	$\nabla^2\varphi = -\frac{\varrho}{\epsilon}$ $\nabla^2\boldsymbol{A} = -\mu\boldsymbol{J}$	$\nabla^2\varphi = -4\pi\frac{\varrho}{\epsilon}$ $\nabla^2\boldsymbol{A} = -\frac{4\pi}{c}\mu\boldsymbol{J}$

Name of formula	SI (MKSA)	Gaussian
Wave equations	$\nabla^2 \boldsymbol{E} - \epsilon\mu \frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0$	$\nabla^2 \boldsymbol{E} - \frac{\epsilon\mu}{c^2} \frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0$
	$\nabla^2 \boldsymbol{H} - \epsilon\mu \frac{\partial^2 \boldsymbol{H}}{\partial t^2} = 0$	$\nabla^2 \boldsymbol{H} - \frac{\epsilon\mu}{c^2} \frac{\partial^2 \boldsymbol{H}}{\partial t^2} = 0$
Dynamic potentials	$\boldsymbol{B} = \nabla \times \boldsymbol{A}$	$\boldsymbol{B} = \nabla \times \boldsymbol{A}$
	$\boldsymbol{E} = -\nabla\varphi - \frac{\partial \boldsymbol{A}}{\partial t}$	$\boldsymbol{E} = -\nabla\varphi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}$
Lorentz gauge	$\nabla \cdot \boldsymbol{A} + \epsilon\mu \frac{\partial \varphi}{\partial t} = 0$	$\nabla \cdot \boldsymbol{A} + \frac{\epsilon\mu}{c} \frac{\partial \varphi}{\partial t} = 0$
D'Alembert equations	$\nabla^2\varphi - \epsilon\mu \frac{\partial^2 \varphi}{\partial t^2} = -\frac{\varrho}{\epsilon}$	$\nabla^2\varphi - \frac{\epsilon\mu}{c^2} \frac{\partial^2 \varphi}{\partial t^2} = -4\pi\frac{\varrho}{\epsilon}$
	$\nabla^2\boldsymbol{A} - \epsilon\mu \frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\mu\boldsymbol{J}$	$\nabla^2\boldsymbol{A} - \frac{\epsilon\mu}{c^2} \frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\frac{4\pi}{c}\mu\boldsymbol{J}$
Retarding potentials	$\varphi = \frac{1}{4\pi\epsilon}\int_V \frac{\varrho(t - r/c)}{r} dV'$	$\varphi = \frac{1}{\epsilon}\int_V \frac{\varrho(t - r/c)}{r} dV'$
	$\boldsymbol{A} = \frac{\mu}{4\pi}\int_V \frac{\boldsymbol{J}(t - r/c)}{r} dV'$	$\boldsymbol{A} = \frac{\mu}{c}\int_V \frac{\boldsymbol{J}(t - r/c)}{r} dV'$
Energy density	$w = \frac{1}{2}(\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{H}\cdot\boldsymbol{B})$	$w = \frac{1}{8\pi}(\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{H}\cdot\boldsymbol{B})$
Poynting vector	$\boldsymbol{P} = \boldsymbol{E} \times \boldsymbol{H}$	$\boldsymbol{P} = \frac{c}{4\pi}\boldsymbol{E} \times \boldsymbol{H}$

A.3 Prefixes and Symbols for Multiples

Multiple	Prefix	Symbol
10^{-18}	atto	a
10^{-15}	femto	f
10^{-12}	pico	p
10^{-9}	nano	n
10^{-6}	micro	μ
10^{-3}	milli	m
10^{-2}	centi	c
10^{-1}	deci	d
10	deka	da
10^{2}	hecto	h
10^{3}	kilo	k
10^{6}	mega	M
10^{9}	giga	G
10^{12}	tera	T
10^{15}	peta	P
10^{18}	exa	E

Appendix B

Vector Analysis

B.1 Vector Differential Operations

B.1.1 General Orthogonal Coordinates

$$u_1,\ u_2,\ u_3,\quad h_1,\ h_2,\ h_3,\quad h_i = \sqrt{\left(\frac{\partial x}{\partial u_i}\right)^2 + \left(\frac{\partial y}{\partial u_i}\right)^2 + \left(\frac{\partial z}{\partial u_i}\right)^2},\ i = 1, 2, 3$$

$$\boldsymbol{A} = \hat{\boldsymbol{u}}_1 A_1 + \hat{\boldsymbol{u}}_2 A_2 + \hat{\boldsymbol{u}}_3 A_3$$

$$\nabla\varphi = \sum_{i=1}^{3} \hat{\boldsymbol{u}}_i \frac{1}{h_i}\frac{\partial \varphi}{\partial u_i} = \hat{\boldsymbol{u}}_1 \frac{1}{h_1}\frac{\partial \varphi}{\partial u_1} + \hat{\boldsymbol{u}}_2 \frac{1}{h_2}\frac{\partial \varphi}{\partial u_2} + \hat{\boldsymbol{u}}_3 \frac{1}{h_3}\frac{\partial \varphi}{\partial u_3} \tag{B.1}$$

$$\begin{aligned}\nabla \cdot \boldsymbol{A} &= \frac{1}{h_1 h_2 h_3} \sum_{i=1}^{3} \frac{\partial}{\partial u_i}(h_j h_k A_i) \\ &= \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 A_1) + \frac{\partial}{\partial u_2}(h_3 h_1 A_2) + \frac{\partial}{\partial u_3}(h_1 h_2 A_3)\right]\end{aligned} \tag{B.2}$$

$$\begin{aligned}\nabla \times \boldsymbol{A} &= \sum_{i=1}^{3} \hat{\boldsymbol{u}}_i \frac{1}{h_j h_k}\left[\frac{\partial}{\partial u_j}(h_k A_k) - \frac{\partial}{\partial u_k}(h_j A_j)\right] \\ &= \hat{\boldsymbol{u}}_1 \frac{1}{h_2 h_3}\left[\frac{\partial}{\partial u_2}(h_3 A_3) - \frac{\partial}{\partial u_3}(h_2 A_2)\right] \\ &\quad + \hat{\boldsymbol{u}}_2 \frac{1}{h_3 h_1}\left[\frac{\partial}{\partial u_3}(h_1 A_1) - \frac{\partial}{\partial u_1}(h_3 A_3)\right] \\ &\quad + \hat{\boldsymbol{u}}_3 \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial u_1}(h_2 A_2) - \frac{\partial}{\partial u_2}(h_1 A_1)\right]\end{aligned} \tag{B.3}$$

$$\nabla^2\varphi = \frac{1}{h_1h_2h_3}\sum_{i=1}^{3}\frac{\partial}{\partial u_i}\Big(\frac{h_jh_k}{h_i}\frac{\partial\varphi}{\partial u_i}\Big)$$
$$= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\Big(\frac{h_2h_3}{h_1}\frac{\partial\varphi}{\partial u_1}\Big) + \frac{\partial}{\partial u_2}\Big(\frac{h_3h_1}{h_2}\frac{\partial\varphi}{\partial u_2}\Big) + \frac{\partial}{\partial u_3}\Big(\frac{h_1h_2}{h_3}\frac{\partial\varphi}{\partial u_3}\Big)\right] \quad \text{(B.4)}$$

$$\begin{aligned}\nabla^2\boldsymbol{A} &= \nabla(\nabla\cdot\boldsymbol{A}) - \nabla\times\nabla\times\boldsymbol{A}\\ &= \hat{\boldsymbol{u}}_1\left[\frac{1}{h_1}\frac{\partial F_0}{\partial u_1} - \frac{1}{h_2h_3}\Big(\frac{\partial F_3}{\partial u_2} - \frac{\partial F_2}{\partial u_3}\Big)\right]\\ &+ \hat{\boldsymbol{u}}_2\left[\frac{1}{h_2}\frac{\partial F_0}{\partial u_2} - \frac{1}{h_3h_1}\Big(\frac{\partial F_1}{\partial u_3} - \frac{\partial F_3}{\partial u_1}\Big)\right]\\ &+ \hat{\boldsymbol{u}}_3\left[\frac{1}{h_3}\frac{\partial F_0}{\partial u_3} - \frac{1}{h_1h_2}\Big(\frac{\partial F_2}{\partial u_1} - \frac{\partial F_1}{\partial u_3}\Big)\right] \qquad \text{(B.5)}\end{aligned}$$

where

$$\begin{aligned}F_0 &= \nabla\cdot\boldsymbol{A}\\ F_1 = h_1(\nabla\times\boldsymbol{A})_1 &= \frac{h_1}{h_2h_3}\left[\frac{\partial}{\partial u_2}(h_3A_3) - \frac{\partial}{\partial u_3}(h_2A_2)\right]\\ F_2 = h_2(\nabla\times\boldsymbol{A})_2 &= \frac{h_2}{h_3h_1}\left[\frac{\partial}{\partial u_3}(h_1A_1) - \frac{\partial}{\partial u_1}(h_3A_3)\right]\\ F_3 = h_3(\nabla\times\boldsymbol{A})_3 &= \frac{h_3}{h_1h_2}\left[\frac{\partial}{\partial u_1}(h_2A_2) - \frac{\partial}{\partial u_2}(h_1A_1)\right]\end{aligned}$$

B.1.2 General Cylindrical Coordinates

$$u_1,\ u_2,\ z,\quad h_3 = 1,\ \frac{\partial h_1}{\partial z} = 0,\ \frac{\partial h_2}{\partial z} = 0$$
$$\nabla\varphi = \hat{\boldsymbol{u}}_1\frac{1}{h_1}\frac{\partial\varphi}{\partial u_1} + \hat{\boldsymbol{u}}_2\frac{1}{h_2}\frac{\partial\varphi}{\partial u_2} + \hat{\boldsymbol{z}}\frac{\partial\varphi}{\partial z} \qquad \text{(B.6)}$$
$$\nabla\cdot\boldsymbol{A} = \frac{1}{h_1h_2}\left[\frac{\partial}{\partial u_1}(h_2A_1) + \frac{\partial}{\partial u_2}(h_1A_2)\right] + \frac{\partial A_z}{\partial z} \qquad \text{(B.7)}$$

$$\begin{aligned}\nabla\times\boldsymbol{A} &= \hat{\boldsymbol{u}}_1\frac{1}{h_2}\left[\frac{\partial A_z}{\partial u_2} - \frac{\partial}{\partial z}(h_2A_2)\right]\\ &+ \hat{\boldsymbol{u}}_2\frac{1}{h_1}\left[\frac{\partial}{\partial z}(h_1A_1) - \frac{\partial A_z}{\partial u_1}\right]\\ &+ \hat{\boldsymbol{u}}_3\frac{1}{h_1h_2}\left[\frac{\partial}{\partial u_1}(h_2A_2) - \frac{\partial}{\partial u_2}(h_1A_1)\right] \qquad \text{(B.8)}\end{aligned}$$

$$\nabla^2\varphi = \frac{1}{h_1h_2}\left[\frac{\partial}{\partial u_1}\Big(\frac{h_2}{h_1}\frac{\partial\varphi}{\partial u_1}\Big) + \frac{\partial}{\partial u_2}\Big(\frac{h_1}{h_2}\frac{\partial\varphi}{\partial u_2}\Big)\right] + \frac{\partial^2\varphi}{\partial z^2} \qquad \text{(B.9)}$$

$$\nabla^2 \boldsymbol{A} = \nabla^2 \boldsymbol{A}_T + \hat{\boldsymbol{z}} \nabla^2 A_z \tag{B.10}$$

where A_z is the longitudinal component and $\boldsymbol{A}_T$ is the transverse 2-dimensional vector of $\boldsymbol{A}$

$$\boldsymbol{A} = \boldsymbol{A}_T + \hat{\boldsymbol{z}} A_z, \qquad \boldsymbol{A}_T = \hat{\boldsymbol{u}}_1 A_1 + \hat{\boldsymbol{u}}_2 A_2$$

$$\nabla^2 A_z = \frac{1}{h_1 h_2} \left[\frac{\partial}{\partial u_1} \left(\frac{h_2}{h_1} \frac{\partial A_z}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_1}{h_2} \frac{\partial A_z}{\partial u_2} \right) \right] + \frac{\partial^2 A_z}{\partial z^2}$$

$$\begin{aligned} \nabla^2 \boldsymbol{A}_T = \hat{\boldsymbol{u}}_1 & \left(\frac{1}{h_1} \frac{\partial F_0}{\partial u_1} - \frac{1}{h_2} \frac{\partial F_z}{\partial u_2} + \frac{\partial^2 A_1}{\partial z^2} \right) \\ + \hat{\boldsymbol{u}}_2 & \left(\frac{1}{h_2} \frac{\partial F_0}{\partial u_2} + \frac{1}{h_1} \frac{\partial F_z}{\partial u_1} + \frac{\partial^2 A_2}{\partial z^2} \right) \end{aligned} \tag{B.11}$$

where

$$F_0 = \nabla \cdot \boldsymbol{A}_T = \frac{1}{h_1 h_2} \left[\frac{\partial}{\partial u_1} (h_2 A_1) + \frac{\partial}{\partial u_2} (h_1 A_2) \right]$$

$$F_z = |\nabla \times \boldsymbol{A}_T| = \frac{1}{h_1 h_2} \left[\frac{\partial}{\partial u_1} (h_2 A_2) - \frac{\partial}{\partial u_2} (h_1 A_1) \right]$$

B.1.3 Rectangular Coordinates

$$x, \; y, \; z, \qquad h_1 = 1, \; h_2 = 1, \; h_3 = 1$$

$$\nabla \varphi = \hat{\boldsymbol{x}} \frac{\partial \varphi}{\partial x} + \hat{\boldsymbol{y}} \frac{\partial \varphi}{\partial y} + \hat{\boldsymbol{z}} \frac{\partial \varphi}{\partial z} \tag{B.12}$$

$$\nabla \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{B.13}$$

$$\nabla \times \boldsymbol{A} = \hat{\boldsymbol{x}} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \hat{\boldsymbol{y}} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \hat{\boldsymbol{z}} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \tag{B.14}$$

$$\nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} \tag{B.15}$$

$$\nabla^2 \boldsymbol{A} = \hat{\boldsymbol{x}} \nabla^2 A_x + \hat{\boldsymbol{y}} \nabla^2 A_y + \hat{\boldsymbol{z}} \nabla^2 A_z \tag{B.16}$$

B.1.4 Circular Cylindrical Coordinates

$$\rho, \; \phi, \; z, \qquad h_1 = 1, \; h_2 = r, \; h_3 = 1$$

$$\nabla \varphi = \hat{\boldsymbol{\rho}} \frac{\partial \varphi}{\partial \rho} + \hat{\boldsymbol{\phi}} \frac{1}{\rho} \frac{\partial \varphi}{\partial \phi} + \hat{\boldsymbol{z}} \frac{\partial \varphi}{\partial z} \tag{B.17}$$

$$\nabla \cdot \boldsymbol{A} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_\rho) + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \tag{B.18}$$

$$\nabla\times\boldsymbol{A}=\hat{\boldsymbol{\rho}}\left[\frac{1}{\rho}\frac{\partial A_z}{\partial\phi}-\frac{\partial A_\phi}{\partial z}\right]+\hat{\boldsymbol{\phi}}\left[\frac{\partial A_\rho}{\partial z}-\frac{\partial A_z}{\partial\rho}\right]+\hat{\boldsymbol{z}}\frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho A_\phi)-\frac{\partial A_\rho}{\partial\phi}\right] \tag{B.19}$$

$$\nabla^2\varphi=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\varphi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\varphi}{\partial\phi^2}+\frac{\partial^2\varphi}{\partial z^2} \tag{B.20}$$

$$\nabla^2\boldsymbol{A}=\hat{\boldsymbol{\rho}}\left(\nabla^2A_\rho-\frac{2}{\rho^2}\frac{\partial A_\phi}{\partial\phi}-\frac{A_\rho}{\rho^2}\right)+\hat{\boldsymbol{\phi}}\left(\nabla^2A_\phi+\frac{2}{\rho^2}\frac{\partial A_\rho}{\partial\phi}-\frac{A_\phi}{\rho^2}\right)+\hat{\boldsymbol{z}}\nabla^2A_z \tag{B.21}$$

B.1.5 Spherical Coordinates

$$r,\ \theta,\ \phi,\qquad h_1=1,\ h_2=r,\ h_3=r\sin\theta$$

$$\nabla\varphi=\hat{\boldsymbol{r}}\frac{\partial\varphi}{\partial r}+\hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial\varphi}{\partial\theta}+\hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial\varphi}{\partial\phi} \tag{B.22}$$

$$\nabla\cdot\boldsymbol{A}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta)+\frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \tag{B.23}$$

$$\begin{aligned}\nabla\times\boldsymbol{A}=\hat{\boldsymbol{r}}&\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi)-\frac{\partial A_\theta}{\partial\phi}\right]\\&+\hat{\boldsymbol{\theta}}\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi}-\frac{\partial}{\partial r}(rA_\phi)\right]+\hat{\boldsymbol{\phi}}\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta)-\frac{\partial A_r}{\partial\theta}\right]\end{aligned} \tag{B.24}$$

$$\nabla^2\varphi=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\varphi}{\partial\phi^2} \tag{B.25}$$

$$\begin{aligned}\nabla^2\boldsymbol{A}=\hat{\boldsymbol{r}}&\left[\nabla^2A_r-\frac{2}{r^2}\left(A_r+\cot\theta A_\theta+\csc\theta\frac{\partial A_\phi}{\partial\phi}+\frac{\partial A_\theta}{\partial\theta}\right)\right]\\&+\hat{\boldsymbol{\theta}}\left[\nabla^2A_\theta-\frac{1}{r^2}\left(\csc^2\theta A_\theta-2\frac{\partial A_r}{\partial\theta}+2\cot\theta\csc\theta\frac{\partial A_\phi}{\partial\phi}\right)\right]\\&+\hat{\boldsymbol{\phi}}\left[\nabla^2A_\phi-\frac{1}{r^2}\left(\csc^2\theta A_\phi-2\csc\theta\frac{\partial A_r}{\partial\phi}-2\cot\theta\csc\theta\frac{\partial A_\theta}{\partial\phi}\right)\right]\end{aligned} \tag{B.26}$$

B.2 Vector Formulas

B.2.1 Vector Algebraic Formulas

$$\boldsymbol{A}\cdot\boldsymbol{B}=\boldsymbol{B}\cdot\boldsymbol{A} \tag{B.27}$$

$$\boldsymbol{A}\times\boldsymbol{B}=-\boldsymbol{B}\times\boldsymbol{A} \tag{B.28}$$

$$\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C})=\boldsymbol{B}\cdot(\boldsymbol{C}\times\boldsymbol{A})=\boldsymbol{C}\cdot(\boldsymbol{A}\times\boldsymbol{B}) \tag{B.29}$$

$$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C})=(\boldsymbol{A}\cdot\boldsymbol{C})\boldsymbol{B}-(\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{C} \tag{B.30}$$

$$(\boldsymbol{A}\times\boldsymbol{B})\cdot(\boldsymbol{C}\times\boldsymbol{D})=(\boldsymbol{A}\cdot\boldsymbol{C})(\boldsymbol{B}\cdot\boldsymbol{D})-(\boldsymbol{A}\cdot\boldsymbol{D})(\boldsymbol{B}\cdot\boldsymbol{C}) \tag{B.31}$$

$$(\boldsymbol{A}\times\boldsymbol{B})\times(\boldsymbol{C}\times\boldsymbol{D})=(\boldsymbol{A}\times\boldsymbol{B}\cdot\boldsymbol{D})\boldsymbol{C}-(\boldsymbol{A}\times\boldsymbol{B}\cdot\boldsymbol{C})\boldsymbol{D} \tag{B.32}$$

B.2.2 Vector Differential Formulas

$$\nabla(\varphi + \psi) = \nabla\varphi + \nabla\psi \tag{B.33}$$

$$\nabla(\varphi\psi) = \varphi\nabla\psi + \psi\nabla\varphi \tag{B.34}$$

$$\nabla(\boldsymbol{A}\cdot\boldsymbol{B}) = (\boldsymbol{A}\cdot\nabla)\boldsymbol{B} + (\boldsymbol{B}\cdot\nabla)\boldsymbol{A} + \boldsymbol{A}\times(\nabla\times\boldsymbol{B}) + \boldsymbol{B}\times(\nabla\times\boldsymbol{A}) \tag{B.35}$$

$$\nabla\cdot(\boldsymbol{A}+\boldsymbol{B}) = \nabla\cdot\boldsymbol{A} + \nabla\cdot\boldsymbol{B} \tag{B.36}$$

$$\nabla\cdot(\varphi\boldsymbol{A}) = \boldsymbol{A}\cdot\nabla\varphi + \varphi\nabla\cdot\boldsymbol{A} \tag{B.37}$$

$$\nabla\cdot(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{B}\cdot(\nabla\times\boldsymbol{A}) - \boldsymbol{A}\cdot(\nabla\times\boldsymbol{B}) \tag{B.38}$$

$$\nabla\times(\boldsymbol{A}+\boldsymbol{B}) = \nabla\times\boldsymbol{A} + \nabla\times\boldsymbol{B} \tag{B.39}$$

$$\nabla\times(\varphi\boldsymbol{A}) = \nabla\varphi\times\boldsymbol{A} + \varphi\nabla\times\boldsymbol{A} \tag{B.40}$$

$$\nabla\times(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{A}(\nabla\cdot\boldsymbol{B}) - \boldsymbol{B}(\nabla\cdot\boldsymbol{A}) + (\boldsymbol{B}\cdot\nabla)\boldsymbol{A} - (\boldsymbol{A}\cdot\nabla)\boldsymbol{B} \tag{B.41}$$

$$\nabla\cdot\nabla\varphi = \nabla^2\varphi \tag{B.42}$$

$$\nabla\times\nabla\varphi = 0 \tag{B.43}$$

$$\nabla\cdot\nabla\times\boldsymbol{A} = 0 \tag{B.44}$$

$$\nabla\times\nabla\times\boldsymbol{A} = \nabla(\nabla\cdot\boldsymbol{A}) - \nabla^2\boldsymbol{A} \tag{B.45}$$

B.2.3 Vector Integral Formulas

Volume V is bounded by closed surface S. The unit vector $\boldsymbol{n}$ is normal to S and directed positively outwards.

$$\int_V \nabla\varphi \mathrm{d}V = \oint_S \varphi\boldsymbol{n}\mathrm{d}S \tag{B.46}$$

$$\int_V \nabla\cdot\boldsymbol{A}\mathrm{d}V = \oint_S \boldsymbol{A}\cdot\boldsymbol{n}\mathrm{d}S \qquad \text{(Gauss's theorem)} \tag{B.47}$$

$$\int_V \nabla\times\boldsymbol{A}\mathrm{d}V = \oint_S \boldsymbol{n}\times\boldsymbol{A}\mathrm{d}S \tag{B.48}$$

$$\int_V (\varphi\nabla^2\psi - \nabla\varphi\nabla\psi)\mathrm{d}V = \oint_S \varphi\nabla\psi\cdot\boldsymbol{n}\mathrm{d}S \qquad \text{(Green's first identity)} \tag{B.49}$$

$$\int_V (\psi\nabla^2\varphi - \varphi\nabla^2\psi)\mathrm{d}V = \oint_S (\psi\nabla\varphi - \varphi\nabla\psi)\cdot\boldsymbol{n}\mathrm{d}S \qquad \text{(Green's second identity)} \tag{B.50}$$

Open surface S is bounded by closed line or contour l.

$$\int_S \boldsymbol{n}\times\nabla\varphi\mathrm{d}S = \oint_l \varphi\mathrm{d}\boldsymbol{l} \tag{B.51}$$

$$\int_S \nabla\times\boldsymbol{A}\cdot\boldsymbol{n}\mathrm{d}S = \oint_l \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{l} \qquad \text{(Stokes's theorem)} \tag{B.52}$$

B.2.4 Differential Formulas for the Position Vector

$$\boldsymbol{x} = \hat{\boldsymbol{x}}x + \hat{\boldsymbol{y}}y + \hat{\boldsymbol{z}}z \qquad \boldsymbol{x}' = \hat{\boldsymbol{x}}x' + \hat{\boldsymbol{y}}y' + \hat{\boldsymbol{z}}z'$$

$$\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}' = \hat{\boldsymbol{r}}r \qquad r = |\boldsymbol{r}| = |\boldsymbol{x} - \boldsymbol{x}'| = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

$$\nabla = \hat{\boldsymbol{x}}\frac{\partial}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z} \qquad \nabla' = \hat{\boldsymbol{x}}\frac{\partial}{\partial x'} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y'} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z'}$$

$$\nabla r = -\nabla' r = \frac{\boldsymbol{r}}{r} = \hat{\boldsymbol{r}} \tag{B.53}$$

$$\nabla \frac{1}{r} = -\nabla' \frac{1}{r} = -\frac{\boldsymbol{r}}{r^3} = -\frac{\hat{\boldsymbol{r}}}{r^2} \tag{B.54}$$

$$\nabla \cdot \frac{\hat{\boldsymbol{r}}}{r^2} = -\nabla \cdot \frac{\hat{\boldsymbol{r}}}{r^2} = 4\pi\delta(\boldsymbol{r}) = 4\pi\delta(\boldsymbol{x} - \boldsymbol{x}') \tag{B.55}$$

$$\nabla^2 \frac{1}{r} = \nabla'^2 \frac{1}{r} = -4\pi\delta(\boldsymbol{r}) = -4\pi\delta(\boldsymbol{x} - \boldsymbol{x}') \tag{B.56}$$

Appendix C

Bessel Functions

C.1 Power Series Representations

Bessel functions of the first kind:

$$\mathrm{J}_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+n)!}\left(\frac{x}{2}\right)^{2m+n} \tag{C.1}$$

Bessel functions of the second kind or Neumann functions:

$$\mathrm{N}_n(x) = \frac{2}{\pi}\ln\frac{\gamma x}{2}\mathrm{J}_n(x) - \frac{1}{\pi}\sum_{m=0}^{n-1}\frac{(n-m-1)!}{m!}\left(\frac{x}{2}\right)^{2m-n}$$

$$-\frac{1}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!(m+n)!}\left(\frac{x}{2}\right)^{2m+n}\left(1+\frac{1}{2}+\cdots+\frac{1}{m}+1+\frac{1}{2}+\cdots+\frac{1}{m+n}\right) \tag{C.2}$$

Modified Bessel functions of the first kind:

$$\mathrm{I}_n(x) = \sum_{m=0}^{\infty}\frac{1}{m!(m+n)!}\left(\frac{x}{2}\right)^{2m+n} \tag{C.3}$$

Modified Bessel functions of the second kind:

$$\mathrm{K}_n(x) = (-1)^{n+1}\ln\frac{\gamma x}{2}\mathrm{I}_n(x) - \frac{1}{2}\sum_{m=0}^{n-1}(-1)^m\frac{(n-m-1)!}{m!}\left(\frac{x}{2}\right)^{2m-n}$$

$$-\frac{(-1)^n}{2}\sum_{m=0}^{\infty}\frac{1}{m!(m+n)!}\left(\frac{x}{2}\right)^{2m+n}\left(1+\frac{1}{2}+\cdots+\frac{1}{m}+1+\frac{1}{2}+\cdots+\frac{1}{m+n}\right) \tag{C.4}$$

where γ is the Euler's constant

$$\ln\gamma = \lim_{n\to\infty}\left(\sum_{m=1}^{n}\frac{1}{m} - \ln n\right), \qquad \ln\gamma = 0.5772, \qquad \gamma = 1.781$$

C.2 Integral Representations

$$\mathrm{J}_n(x)=\frac{\mathrm{j}^{-n}}{2\pi}\int_0^{2\pi}\mathrm{e}^{\mathrm{j}(x\cos\alpha-n\alpha)}\mathrm{d}\alpha,\qquad \mathrm{J}_0(x)=\frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{\mathrm{j}x\cos\alpha}\mathrm{d}\alpha \tag{C.5}$$

C.3 Approximate Expressions

C.3.1 Leading Terms of Power Series (Small Argument)

$$\mathrm{J}_0(x)\xrightarrow{x\to 0}1-\frac{x^2}{4},\qquad \mathrm{J}_1(x)\xrightarrow{x\to 0}\frac{x}{2}-\frac{x^3}{16},\qquad \mathrm{J}_n(x)\xrightarrow{x\to 0}\frac{1}{n!}\left(\frac{x}{2}\right)^n \tag{C.6}$$

$$\mathrm{N}_0(x)\xrightarrow{x\to 0}\frac{2}{\pi}\ln\frac{\gamma x}{2},\quad \mathrm{N}_1(x)\xrightarrow{x\to 0}\frac{2}{\pi x},\quad \mathrm{N}_n(x)\xrightarrow{x\to 0}\frac{(n-1)!}{\pi}\left(\frac{2}{x}\right)^n,\ (n\neq 0) \tag{C.7}$$

$$\mathrm{I}_0(x)\xrightarrow{x\to 0}1+\frac{x^2}{4},\qquad \mathrm{I}_1(x)\xrightarrow{x\to 0}\frac{x}{2}+\frac{x^3}{16},\qquad \mathrm{I}_n(x)\xrightarrow{x\to 0}\frac{1}{n!}\left(\frac{x}{2}\right)^n \tag{C.8}$$

$$\mathrm{K}_0(x)\xrightarrow{x\to 0}\ln\frac{2}{\gamma x},\qquad \mathrm{K}_1(x)\xrightarrow{x\to 0}\frac{1}{x},\qquad \mathrm{K}_n(x)\xrightarrow{x\to 0}\frac{(n-1)!}{2}\left(\frac{2}{x}\right)^n,\ (n\neq 0) \tag{C.9}$$

C.3.2 Leading Terms of Asymptotic Series (Large Argument)

$$\mathrm{J}_n(x)\xrightarrow{x\to\infty}\sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{\pi}{4}-\frac{n\pi}{2}\right),\quad \mathrm{N}_n(x)\xrightarrow{x\to\infty}\sqrt{\frac{2}{\pi x}}\sin\left(x-\frac{\pi}{4}-\frac{n\pi}{2}\right) \tag{C.10}$$

$$\mathrm{H}_n^{(1)}(x)\xrightarrow{x\to\infty}\sqrt{\frac{2}{\pi x}}\mathrm{e}^{\mathrm{j}\left(x-\frac{\pi}{4}-\frac{n\pi}{2}\right)},\qquad \mathrm{H}_n^{(2)}(x)\xrightarrow{x\to\infty}\sqrt{\frac{2}{\pi x}}\mathrm{e}^{-\mathrm{j}\left(x-\frac{\pi}{4}-\frac{n\pi}{2}\right)} \tag{C.11}$$

$$\mathrm{I}_n(x)\xrightarrow{x\to\infty}\frac{1}{\sqrt{2\pi x}}\mathrm{e}^{x},\qquad \mathrm{K}_n(x)\xrightarrow{x\to\infty}\sqrt{\frac{\pi}{2x}}\mathrm{e}^{-x} \tag{C.12}$$

C.4 Formulas for Bessel Functions

$\mathrm{Z}_n(x)$ represents $\mathrm{J}_n(x)$, $\mathrm{N}_n(x)$, $\mathrm{H}_n^{(1)}(x)$, and $\mathrm{H}_n^{(2)}(x)$.

C.4.1 Recurrence Formulas

$$\mathrm{Z}_{n-1}(x)+\mathrm{Z}_{n+1}(x)=\frac{2n}{x}\mathrm{Z}_n(x) \tag{C.13}$$

$$\mathrm{I}_{n-1}(x)-\mathrm{I}_{n+1}(x)=\frac{2n}{x}\mathrm{I}_n(x) \tag{C.14}$$

$$\mathrm{K}_{n-1}(x)-\mathrm{K}_{n+1}(x)=-\frac{2n}{x}\mathrm{K}_n(x) \tag{C.15}$$

C.4.2 Derivatives

$$\mathrm{Z}_n'(x)=\frac{1}{2}[\mathrm{Z}_{n-1}(x)-\mathrm{Z}_{n+1}(x)]=\mathrm{Z}_{n-1}(x)-\frac{n}{x}\mathrm{Z}_n(x)=\frac{n}{x}\mathrm{Z}_n(x)-\mathrm{Z}_{n+1}(x) \tag{C.16}$$

$$\mathrm{I}_n'(x)=\frac{1}{2}[\mathrm{I}_{n-1}(x)+\mathrm{I}_{n+1}(x)]=\mathrm{I}_{n-1}(x)-\frac{n}{x}\mathrm{I}_n(x)=\frac{n}{x}\mathrm{I}_n(x)+\mathrm{I}_{n+1}(x) \tag{C.17}$$

$$\mathrm{K}_n'(x)=-\frac{1}{2}[\mathrm{K}_{n-1}(x)+\mathrm{K}_{n+1}(x)]=-\mathrm{K}_{n-1}(x)-\frac{n}{x}\mathrm{K}_n(x)=\frac{n}{x}\mathrm{K}_n(x)-\mathrm{K}_{n+1}(x) \tag{C.18}$$

$$\mathrm{Z}_0'(x)=-\mathrm{Z}_1(x), \qquad \mathrm{I}_0'(x)=\mathrm{I}_1(x), \qquad \mathrm{K}_0'(x)=-\mathrm{K}_1(x) \tag{C.19}$$

$$\mathrm{Z}_1'(x)=\mathrm{Z}_0(x)-\frac{1}{x}\mathrm{Z}_1(x),\ \ \mathrm{I}_1'(x)=\mathrm{I}_0(x)-\frac{1}{x}\mathrm{I}_1(x),\ \ \mathrm{K}_0'(x)=-\mathrm{K}_0(x)-\frac{1}{x}\mathrm{K}_1(x) \tag{C.20}$$

C.4.3 Integrals

$$\int x^{n+1}\mathrm{Z}_n(x)\mathrm{d}x=x^{n+1}\mathrm{Z}_{n+1}(x) \tag{C.21}$$

$$\int x^{-(n-1)}\mathrm{Z}_n(x)\mathrm{d}x=-x^{-(n-1)}\mathrm{Z}_{n-1}(x) \tag{C.22}$$

$$\int x\mathrm{Z}_n^2(kx)\mathrm{d}x=\frac{x^2}{2}[\mathrm{Z}_n^2(kx)-\mathrm{Z}_{n-1}(kx)\mathrm{Z}_{n+1}(kx)] \tag{C.23}$$

C.4.4 Wronskian

Define $W[F_1(x),F_2(x)]=F_1(x)\dfrac{\mathrm{d}F_2(x)}{\mathrm{d}x}-F_2(x)\dfrac{\mathrm{d}F_1(x)}{\mathrm{d}x}$ as the *Wronskian*.

$$W[\mathrm{J}_n(x),\mathrm{N}_n(x)]=\frac{2}{\pi x} \tag{C.24}$$

$$W[\mathrm{J}_n(x),\mathrm{H}_n^{(1)}(x)]=\mathrm{j}\frac{2}{\pi x} \tag{C.25}$$

$$W[\mathrm{J}_n(x),\mathrm{H}_n^{(2)}(x)]=-\mathrm{j}\frac{2}{\pi x} \tag{C.26}$$

$$W[\mathrm{H}_n^{(1)}(x),\mathrm{H}_n^{(2)}(x)]=\mathrm{j}\frac{4}{\pi x} \tag{C.27}$$

$$W[\mathrm{I}_n(x),\mathrm{K}_n(x)]=-\frac{1}{x} \tag{C.28}$$

Consequently

$$\mathrm{J}_n(x)\mathrm{N}_{n+1}(x)-\mathrm{N}_n(x)\mathrm{J}_{n+1}(x)=-\frac{2}{\pi x} \tag{C.29}$$

$$\mathrm{I}_n(x)\mathrm{K}_{n+1}(x)-\mathrm{K}_n(x)\mathrm{I}_{n+1}(x)=\frac{1}{x} \tag{C.30}$$

C.5 Spherical Bessel Functions

C.5.1 Bessel Functions of Order $n+1/2$

$$\mathrm{J}_{n+1/2}(x) = \sqrt{\frac{2}{\pi}}x^{n+1/2}\Big(-\frac{1}{x}\Big)^n \frac{\mathrm{d}^n}{\mathrm{d}x^n}\frac{\sin x}{x} \tag{C.31}$$

$$\mathrm{N}_{n+1/2}(x) = (-1)^{n+1}\sqrt{\frac{2}{\pi}}x^{n+1/2}\Big(\frac{1}{x}\Big)^n \frac{\mathrm{d}^n}{\mathrm{d}x^n}\frac{\cos x}{x} \tag{C.32}$$

$$\mathrm{H}^{(1)}_{n+1/2}(x) = \sqrt{\frac{2}{\pi x}}\mathrm{e}^{-\mathrm{j}n(n+1)/2}\mathrm{e}^{\mathrm{j}x}\sum_{k=0}^{n}(-1)^k\frac{(n+k)!}{k!(n-k)!}\frac{1}{(2\mathrm{j}x)^k} \tag{C.33}$$

$$\mathrm{H}^{(2)}_{n+1/2}(x) = \sqrt{\frac{2}{\pi x}}\mathrm{e}^{\mathrm{j}n(n+1)/2}\mathrm{e}^{-\mathrm{j}x}\sum_{k=0}^{n}\frac{(n+k)!}{k!(n-k)!}\frac{1}{(2\mathrm{j}x)^k} \tag{C.34}$$

$$\mathrm{J}_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x, \qquad \mathrm{J}_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\Big(\frac{\sin x}{x} - \cos x\Big) \tag{C.35}$$

$$\mathrm{N}_{1/2}(x) = -\sqrt{\frac{2}{\pi x}}\cos x, \qquad \mathrm{N}_{3/2}(x) = -\sqrt{\frac{2}{\pi x}}\Big(\sin x + \frac{\cos x}{x} - \Big) \tag{C.36}$$

$$\mathrm{H}^{(1)}_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\frac{\mathrm{e}^{\mathrm{j}x}}{\mathrm{j}}, \qquad \mathrm{H}^{(1)}_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\Big(\frac{\mathrm{e}^{\mathrm{j}x}}{\mathrm{j}x} - \mathrm{e}^{\mathrm{j}x}\Big) \tag{C.37}$$

$$\mathrm{H}^{(2)}_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\frac{\mathrm{e}^{-\mathrm{j}x}}{-\mathrm{j}}, \qquad \mathrm{H}^{(2)}_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\Big(\frac{\mathrm{e}^{-\mathrm{j}x}}{-\mathrm{j}x} - \mathrm{e}^{-\mathrm{j}x}\Big) \tag{C.38}$$

C.5.2 Spherical Bessel Functions

$$\mathrm{j}_n(x) = \sqrt{\frac{\pi}{2x}}\mathrm{J}_{n+1/2}(x), \qquad \mathrm{n}_n(x) = \sqrt{\frac{\pi}{2x}}\mathrm{N}_{n+1/2}(x) \tag{C.39}$$

$$\mathrm{h}^{(1)}_n(x) = \sqrt{\frac{\pi}{2x}}\mathrm{H}^{(1)}_{n+1/2}(x), \qquad \mathrm{h}^{(2)}_n(x) = \sqrt{\frac{\pi}{2x}}\mathrm{H}^{(2)}_{n+1/2}(x) \tag{C.40}$$

C.5.3 Spherical Bessel Functions by S.A.Schelkunoff

$$\hat{\mathrm{J}}_n(x) = \sqrt{\frac{\pi x}{2}}\mathrm{J}_{n+1/2}(x), \qquad \hat{\mathrm{N}}_n(x) = \sqrt{\frac{\pi x}{2}}\mathrm{N}_{n+1/2}(x) \tag{C.41}$$

$$\hat{\mathrm{H}}^{(1)}_n(x) = \sqrt{\frac{\pi x}{2}}\mathrm{H}^{(1)}_{n+1/2}(x), \qquad \hat{\mathrm{H}}^{(2)}_n(x) = \sqrt{\frac{\pi x}{2}}\mathrm{H}^{(2)}_{n+1/2}(x) \tag{C.42}$$

Appendix D

Legendre Functions

D.1 Legendre Polynomials

$$\mathrm{P}_n(x) = \frac{1}{2^n\, n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^2-1\right)^n, \tag{D.1}$$

$$\mathrm{Q}_n(x) = \frac{1}{2}\mathrm{P}_n(x)\ln\frac{1+x}{1-x} - \sum_{l=1}^{n}\frac{1}{l}\mathrm{P}_{l-1}(x)\mathrm{P}n-1(x). \tag{D.2}$$

$$\mathrm{P}_0(x)=1, \qquad \mathrm{Q}_0(x)=\frac{1}{2}\ln\frac{1+x}{1-x} \tag{D.3}$$

$$\mathrm{P}_1(x)=x, \qquad \mathrm{Q}_1(x)=x\mathrm{Q}_0(x)-1 \tag{D.4}$$

$$\mathrm{P}_2(x)=\frac{1}{2}(3x^2-1), \qquad \mathrm{Q}_2(x)=\mathrm{P}_2(x)\mathrm{Q}_0(x)-\frac{3}{2}x \tag{D.5}$$

$$\mathrm{P}_3(x)=\frac{1}{2}(5x^3-3x), \qquad \mathrm{Q}_3(x)=\mathrm{P}_3(x)\mathrm{Q}_0(x)-\frac{5}{2}x^2+\frac{3}{2} \tag{D.6}$$

$$\mathrm{P}_4(x)=\frac{1}{8}(35x^4-30x^2+3), \qquad \mathrm{Q}_4(x)=\mathrm{P}_4(x)\mathrm{Q}_0(x)-\frac{35}{8}x^3+\frac{55}{24}x \tag{D.7}$$

D.2 Associate Legendre Polynomials

$$\mathrm{P}_n^m(x)=\left(x^2-1\right)^{m/2}\frac{\mathrm{d}^m}{\mathrm{d}x^m}\mathrm{P}_n(x) \qquad \mathrm{Q}_n^m(x)=\left(x^2-1\right)^{m/2}\frac{\mathrm{d}^m}{\mathrm{d}x^m}\mathrm{Q}_n(x). \tag{D.8}$$

$$\mathrm{P}_1^1(x) = (1-x^2)^{1/2} \tag{D.9}$$

$$\mathrm{P}_2^1(x) = 3(1-x^2)^{1/2}x \tag{D.10}$$

$$\mathrm{P}_3^1(x) = \frac{3}{2}(1-x^2)^{1/2}(5x^2-1) \tag{D.11}$$

$$\mathrm{P}_4^1(x) = \frac{5}{2}(1-x^2)^{1/2}(7x^3-3x) \tag{D.12}$$

$$P_2^2(x) = 3(1-x^2) \tag{D.13}$$
$$P_3^2(x) = 15(1-x^2)x \tag{D.14}$$
$$P_4^2(x) = \frac{15}{2}(1-x^2)(7x^2-1) \tag{D.15}$$

$$P_3^3(x) = 15(1-x^2)^{3/2} \tag{D.16}$$
$$P_4^3(x) = 105(1-x^2)^{3/2}x \tag{D.17}$$

$$P_4^4(x) = 105(1-x^2)^2 \tag{D.18}$$

D.3 Formulas for Legendre Polynomials

In the following formulas, $R_n(x)$ represents $P_n(x)$ and $Q_n(x)$, $R_n^m(x)$ represents $P_n^m(x)$ and $Q_n^m(x)$ including $m = 0$.

D.3.1 Recurrence Formulas

$$(2n+1)xR_n^m(x) = (n+m)R_{n-1}^m(x) + (n-m+1)R_{n+1}^m(x) \tag{D.19}$$
$$2m\frac{x}{\sqrt{1-x^2}}R_n^m(x) = (n-m-1)(n+m)R_n^{m-1}(x) + R_n^{m+1}(x) \tag{D.20}$$

D.3.2 Derivatives

$$(2n+1)R_n(x) = R'_{n+1}(x) - R'_{n-1}(x) \tag{D.21}$$
$$(x^2-1)R_n^{m\prime}(x) = (n-m+1)R_{n+1}^m(x) - (n+1)xR_n^m(x) \tag{D.22}$$

D.3.3 Integrals

$$\int R_n(x)\mathrm{d}x = \frac{R_{n+1}(x) - R_{n-1}(x)}{2n+1} \tag{D.23}$$
$$\int_{-1}^{+1} P_n(x)P_l(x)\mathrm{d}x = 0, \quad \text{for} \quad n \neq l \tag{D.24}$$
$$\int_{-1}^{+1} [P_n(x)]^2\mathrm{d}x = \frac{2}{2n+1}, \tag{D.25}$$
$$\int_{-1}^{+1} P_n^m(x)P_l^m(x)\mathrm{d}x = 0, \quad \text{for} \quad n \neq l \tag{D.26}$$
$$\int_{-1}^{+1} [P_n^m(x)]^2\mathrm{d}x = \frac{2}{(2n+1)}\frac{(n+m)!}{(n-m)!}, \tag{D.27}$$

Appendix E

Matrices and Tensors

E.1 Matrix

1. A *matrix* (a) or $\mathbf{a}$ is a rectangular array of real or complex scalars a_{ij} which are called the elements of the matrix,

$$\mathbf{a} = (a) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} = (a_{ij})_{mn} \qquad \text{(E.1)}$$

2. A *square matrix* is a matrix with $m = n$.

3. A *row matrix* is a matrix with $m = 1$ and a *Column matrix* is a matrix with $n = 1$.

4. The *determinant* of a square matrix is a determinant consisting of the elements of the matrix, and is denoted by $|A|$ or $|(A)|$.

5. The *complementary minor* is the determinant of the sub-matrix obtained from the square matrix (A) by deleting the ith row and the jth column, and is denoted by M_{ij} .

6. The *algebraic complement* A_{ij} is defined as

$$A_{ij} = (-1)^{i+j} M_{ij}$$

7. A *diagonal matrix* is a square matrix whose off-diagonal elements are zero, i.e., $a_{ij} = 0$, $i \neq j$.

8. A *unit matrix* $(\mathcal{I})$ is a diagonal matrix where $a_{ii} = 1$; consequently, $|(\mathcal{I})| = 1$.

9. A *zero matrix* (0) is the matrix where $a_{ij} = 0$, for all i, j.

E.2 Matrix Algebra

E.2.1 Definitions

1. Addition. The sum of two matrices exists only if the two matrices have the same size, i.e., the same number of rows and columns,

$$(A) + (B) = (a_{ij})_{mn} + (b_{ij})_{mn} = (a_{ij} + b_{ij})_{mn} \tag{E.2}$$

2. Multiplication. If α is a scalar

$$\alpha(A) = (\alpha a_{ij})_{mn} \tag{E.3}$$

3. Product of matrices. The product of two matrices exists only if the number of columns in (A) equals the number of rows in (B), i.e., $(A) = (a_{ij})_{mp}$, $(B) = (b_{ij})_{pn}$,

$$(A)(B) = (c_{ij})_{mn} \tag{E.4}$$

where $i = 1, 2 \cdots m$, $j = 1, 2 \cdots n$, and

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{ip} b_{pj}$$

E.2.2 Matrix Algebraic Formulas

In the following expressions, $(A) = (a_{ij})_{mn}$, $(B) = (b_{ij})_{mn}$, $(C) = (c_{ij})_{mn}$ are matrices and α and β are scalars.

$$(A) + (B) = (B) + (A) \tag{E.5}$$

$$[(A) + (B)] + (C) = (A) + [(B) + (C)] \tag{E.6}$$

$$\alpha[(A) + (B)] = \alpha(A) + \alpha(B) \tag{E.7}$$

$$(\alpha + \beta)(A) = \alpha(A) + \beta(A) \tag{E.8}$$

$$(A)(B) \neq (B)(A) \tag{E.9}$$

$$\alpha[(A)(B)] = [\alpha(A)](B) = (A)[\alpha(B)] \tag{E.10}$$

$$[(A)(B)](C) = (A)[(B)(C)] \tag{E.11}$$

$$[(A) + (B)](C) = (A)(C) + (B)(C) \tag{E.12}$$

$$(A)[(B)+(C)] = (A)(B)+(A)(C) \tag{E.13}$$

$$|(A)(B)| = |A||B| \tag{E.14}$$

$$|\alpha(A)| = \alpha^n|A| \tag{E.15}$$

$$(A)+(0) = (A) \tag{E.16}$$

$$(0)(A) = (A)(0) = (0) \tag{E.17}$$

$$(\mathcal{I})(A) = (A)(\mathcal{I}) = (A) \tag{E.18}$$

E.3 Matrix Functions

1. The *negative* of a matrix (A) is the matrix obtained by taking the negative value of all elements of (A), and is denoted by $-(A)$,

$$-(A) = (-a_{ij})_{mn}, \quad \text{and} \quad (A)+(-(A)) = 0, \quad -(-(A)) = (A) \tag{E.19}$$

2. The *conjugate* of a matrix (A) is the matrix obtained by taking the conjugate of all elements of (A), and is denoted by $(A)^*$,

$$(A)^* = (a^*_{ij})_{mn} \tag{E.20}$$

$$((A)^*)^* = (A) \tag{E.21}$$

$$((A)(B))^* = (A)^*(B)^* \tag{E.22}$$

3. The *transpose* of a matrix (A) is the matrix obtained by interchanging the rows of (A) and the columns of (A) and vice versa, and is denoted by $(A)^{\mathrm{T}}$,

$$(A)^{\mathrm{T}} = (a_{ji})_{nm} \tag{E.23}$$

$$((A)^{\mathrm{T}})^{\mathrm{T}} = (A) \tag{E.24}$$

$$((A)(B))^{\mathrm{T}} = (B)^{\mathrm{T}}(A)^{\mathrm{T}} \tag{E.25}$$

$$(A)^{*\mathrm{T}} = (A)^{\mathrm{T}*} \tag{E.26}$$

4. The *conjugate transpose* of a matrix (A) is the matrix obtained by applying the conjugate and transpose operations on (A) simultaneously, and is denoted by $(A)^\dagger$,

$$(A)^\dagger = (A)^{*\mathrm{T}} \tag{E.27}$$

$$((A)(B))^\dagger = (B)^\dagger(A)^\dagger \tag{E.28}$$

5. The *adjoint* of a square matrix (A) is a square matrix whose elements is equal to the elements of the algebraic complement A_{ij} in the matrix (A), and is denoted by $(A)^{\mathrm{a}}$,

$$(A)^{\mathrm{a}} = (A_{ij})_{nn} \tag{E.29}$$

$$(A)(A)^{\mathrm{a}} = |A|(\mathcal{I}) \tag{E.30}$$

6. The *inverse* of a matrix (A) is the matrix that the product of (A) and its inverse matrix is a unit matrix. The matrix (A) has a unique inverse only if it is square and nonsingular, and is denoted by $(A)^{-1}$,

$$(A)(A)^{-1} = (A)^{-1}(A) = (\mathcal{I}) \tag{E.31}$$

$$(A)^{-1} = \frac{1}{|A|}(A)^{\mathrm{a}} = \frac{1}{|A|}(A_{ij})_{nn} \tag{E.32}$$

$$((A)^{-1})^{-1} = (A) \tag{E.33}$$

$$(\alpha(A))^{-1} = \frac{1}{\alpha}(A)^{-1} \tag{E.34}$$

$$((A)(B))^{-1} = (B)^{-1}(A)^{-1} \tag{E.35}$$

$$((A)^*)^{-1} = ((A)^{-1})^*, \quad ((A)^{\mathrm{T}})^{-1} = ((A)^{-1})^{\mathrm{T}}, \quad ((A)^\dagger)^{-1} = ((A)^{-1})^\dagger \tag{E.36}$$

E.4 Special Matrices

1. *Real matrix*,

$$(A)^* = (A), \qquad \text{all } a_{ij} \text{ are real}$$

2. *Symmetric matrix*,

$$(A)^{\mathrm{T}} = (A), \qquad a_{ij} = a_{ji}$$

3. *Skew-symmetric matrix*,

$$(A)^{\mathrm{T}} = -(A), \qquad a_{ij} = 0, \text{ for } i = j, \quad a_{ij} = -a_{ji}, \text{ for } i \neq j$$

4. *Hermitian matrix*,

$$(A)^\dagger = (A), \qquad (A)^{\mathrm{T}} = (A)^*$$

5. *Skew-Hermitian matrix*,

$$(A)^\dagger = -(A), \qquad (A)^{\mathrm{T}} = -(A)^*$$

6. *Unitary matrix* or *U matrix*,

$$(A)^\dagger(A) = (\mathcal{I}), \quad (A)^\dagger = (A)^{-1}, \quad (A) = ((A)^\dagger)^{-1}$$

7. *Orthogonal matrix* is a real unitary matrix,

$$(A)^{\mathrm{T}}(A) = (\mathcal{I}), \quad (A)^{\mathrm{T}} = (A)^{-1}, \quad (A) = ((A)^{\mathrm{T}})^{-1}$$

E.5 Tensors and vectors

1. A *vector* can be expressed by a row matrix or column matrix with three elements, and is denoted by a italic bold-face letter $\boldsymbol{A}$,

$$\boldsymbol{A} = [A_x \ A_y \ A_z] = \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \tag{E.37}$$

2. A *tensor* of rank 2 can be expressed by a 3×3 square matrix and is denoted by a bold face letter $\mathbf{a}$,

$$\mathbf{a} = \begin{bmatrix} a_{xx} & a_{xy} & a_{xz} \\ a_{yx} & a_{yy} & a_{yz} \\ a_{zx} & a_{zy} & a_{zz} \end{bmatrix} \tag{E.38}$$

3. Vector and tensor operations

$$\mathbf{a} \cdot \boldsymbol{A} = \begin{bmatrix} a_{xx} & a_{xy} & a_{xz} \\ a_{yx} & a_{yy} & a_{yz} \\ a_{zx} & a_{zy} & a_{zz} \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \tag{E.39}$$

$$\boldsymbol{A} \cdot \mathbf{a} = [A_x \ A_y \ A_z] \begin{bmatrix} a_{xx} & a_{xy} & a_{xz} \\ a_{yx} & a_{yy} & a_{yz} \\ a_{zx} & a_{zy} & a_{zz} \end{bmatrix} \tag{E.40}$$

$$\mathbf{a} \cdot \mathbf{b} = \begin{bmatrix} a_{xx} & a_{xy} & a_{xz} \\ a_{yx} & a_{yy} & a_{yz} \\ a_{zx} & a_{zy} & a_{zz} \end{bmatrix} \begin{bmatrix} b_{xx} & b_{xy} & b_{xz} \\ b_{yx} & b_{yy} & b_{yz} \\ b_{zx} & b_{zy} & b_{zz} \end{bmatrix} \tag{E.41}$$

$$\boldsymbol{A} \cdot \mathbf{a} \cdot \boldsymbol{A}^* = \boldsymbol{A}^* \cdot \mathbf{a}^{\mathrm{T}} \cdot \boldsymbol{A} \tag{E.42}$$

$$\boldsymbol{A} \cdot \mathbf{a}^* \cdot \boldsymbol{A}^* = \boldsymbol{A}^* \cdot \mathbf{a}^{\dagger} \cdot \boldsymbol{A} \tag{E.43}$$

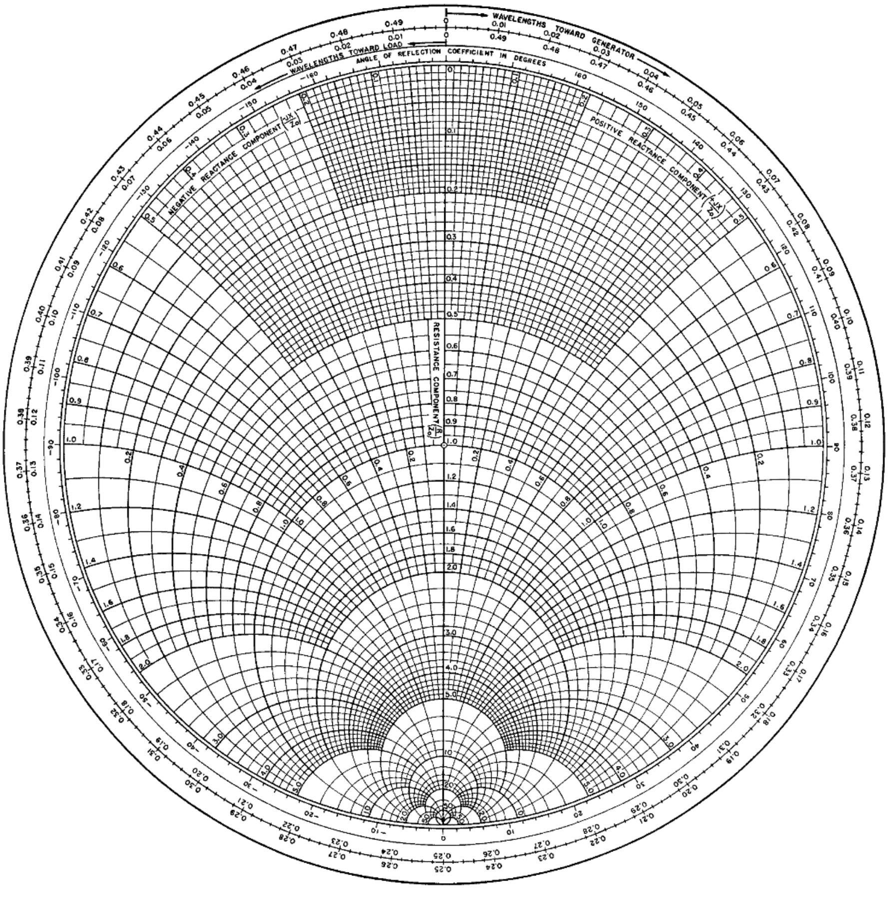
$$\boldsymbol{A} \cdot \mathbf{a} \cdot \boldsymbol{B} = \boldsymbol{B} \cdot \mathbf{a}^{\mathrm{T}} \cdot \boldsymbol{A} \tag{E.44}$$

$$\boldsymbol{A} \cdot \mathbf{a}^* \cdot \boldsymbol{B} = \boldsymbol{B} \cdot \mathbf{a}^{\dagger} \cdot \boldsymbol{A} \tag{E.45}$$

Physical Constants

Physical constant	Symbol	
Speed of light in vacuum	c	299792458 m/s $\approx 3 \times 10^8$ m/s
Vacuum permittivity	ϵ_0	$\frac{1}{4\pi c^2} \times 10^7$ F/m $\approx 8.85418782 \times 10^{-12}$ F/m
Vacuum permeability	μ_0	$4\pi \times 10^{-7}$ H/m $\approx 12.5663706 \times 10^{-7}$ H/m
Vacuum wave impedance	η_0	$4\pi c \times 10^{-7}$ Ω $\approx 120\pi$ Ω
Electron charge magnitude	e	$1.6021892 \times 10^{-19}$ C
Electron rest mass	m	9.109534×10^{-31} kg
Electron charge to mass ratio	e/m	1.75883×10^{11} C/kg
Proton rest mass	m	$1.6726485 \times 10^{-27}$ kg
Gyromagnetic ratio	γ	1.75883×10^{11} rad/(s·T)

Smith Chart



WAVELENGTHS TOWARD GENERATOR
WAVELENGTHS TOWARD LOAD
ANGLE OF REFLECTION COEFFICIENT IN DEGREES
NEGATIVE REACTANCE COMPONENT
POSITIVE REACTANCE COMPONENT
RESISTANCE COMPONENT

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