The number of electrons ΔN that flip their magnetic moment from $-\mu_B$ to μ_B are those contained in the energy interval $\mu_B B$ around the Fermi level:

$$\Delta N = \frac{1}{2} D(E_F) \mu_B B,$$

where $D(E_F)/2 = D_0(E_F)$ is the density-of-states at the Fermi level for one spin direction. The magnetic moment M per unit volume is then

$$M = \frac{1}{V} \Delta N 2\mu_B = \mu_B^2 B \frac{D(E_F)}{V};$$

the ratio M/B gives for the Pauli spin susceptibility the value

$$\boxed{\chi_P = \mu_B^2 \frac{D(E_F)}{V}}. \tag{15.31}$$

In the case of the free-electron gas, the net effect of the Pauli paramagnetic susceptibility (15.31) and the Landau diamagnetic susceptibility (15.28) is a paramagnetic behavior; in fact $\chi_L = -(1/3)\chi_P$. In actual materials, the "effective mass" for orbital motion, as well as the "effective gyromagnetic factor," can be rather different from the corresponding free-electron values, so changing the relative importance of the Pauli and Landau susceptibility; this explains why some metals may have a net diamagnetic behavior. For a more quantitative account, correlation, and exchange effects among electrons should be considered, because they may significantly influence the magnetic susceptibility.

15.5 Magnetoresistivity and Classical Hall Effect

General Considerations and Phenomenological Aspects

Transport effects in crystals in the presence of electric fields and temperature gradients have been considered in Chapter 11. In this section, we study some aspects of transport phenomena due to the simultaneous presence of electric and magnetic fields; the possible presence of thermal gradients adds further variety to the phenomenology, but here we confine our attention to samples at uniform temperature.

The study of magnetic field effects on the transport properties of metals and semiconductors has become a well-established and invaluable tool for the investigation of mobile carriers in crystals. In particular the Hall measurements, aimed at the determination of carrier concentration and charge sign, are routinely used for the characterization of materials. Also magnetoresistivity measurements, which determine the resistivity of materials in the presence of magnetic fields, offer a wide range of effects. In metals with closed Fermi surfaces (such as alkali metals), the magnetoresistivity does saturate for any crystal orientation (i.e. it approaches a constant value for sufficiently high magnetic fields, irrespective of orientation); the same occurs for n-type and p-type semiconductors. In metals with equal number of electrons and holes (such as Bi, Sb,

and so on), the magnetoresistivity does not saturate for any crystal orientation and keeps on increasing as the magnetic field increases; the same occurs for semiconductors with equal numbers of electrons and holes. We also mention that in metals with open Fermi surfaces (such as Cu, Ag, Au, and so on), the magnetoresistivity saturates for most of the crystal orientations but does not saturate for others. Finally, and most importantly, for two-dimensional systems the magnetoresistance is quantized and the quantum Hall effect occurs. In this section we consider some aspects of the traditional magnetoresistivity and Hall phenomenology in three-dimensional crystals, while in the next section we consider the quantized Hall effect in two-dimensional systems.

In isotropic media the application of a (small) electric field drives a current density parallel and proportional to it, and the linear relationship holds

$$\mathbf{J} = \sigma \mathbf{E}, \tag{15.32a}$$

where the conductivity σ is a scalar quantity. In the presence of a magnetic field, carriers are deflected and in general the current density is no more parallel to the electric field; the conductivity becomes a tensor even for an isotropic material. Relation (15.32a) has to be replaced by the more general expression

$$J_i = \sum_j \sigma_{ij}(B) E_j, \tag{15.32b}$$

where $\sigma_{ij}(B)$ $(i, j = x, y, z)$ are the components of the *magnetoconductivity tensor* $\sigma(B)$. Similar considerations can be done for the resistivity of an isotropic medium in the presence of a magnetic field; the relationship between electric field and current density becomes

$$E_i = \sum_j \rho_{ij}(B) J_j, \tag{15.32c}$$

where $\rho_{ij}(B)$ $(i, j = x, y, z)$ are the components of the *magnetoresistivity tensor* $\rho(B)$. The magnetoconductivity tensor and magnetoresistivity tensor are the inverse of each other, and it holds

$$\rho_{ij}(B) = \left(\frac{1}{\sigma(B)} \right)_{ij}. \tag{15.32d}$$

The transport parameters $\rho_{ij}(B)$ are often determined experimentally *using the standard geometry* in which a magnetic field $\mathbf{B}$ is applied orthogonally to a long and thin current carrying conductor and the current flows, for instance, along the x-direction (see Figure 15.9); the x- and y-directions are often referred to as "longitudinal" and "transverse" directions, respectively.

In the standard geometry, in which transport is in the xy plane and furthermore $J_y \equiv 0$ (in stationary conditions), the density current $\mathbf{J}$, and the electric field $\mathbf{E}$ are related by

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} \rho_{xx}(B) & \rho_{xy}(B) \\ \rho_{yx}(B) & \rho_{yy}(B) \end{pmatrix} \begin{pmatrix} J_x \\ J_y \equiv 0 \end{pmatrix}.$$

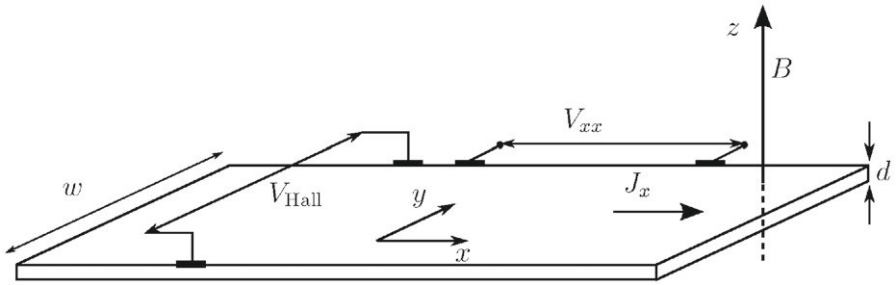

Figure 15.9 Standard geometry for Hall effect and magnetoresistivity measurements. V_{Hall} is the Hall potential, J_x is the current density in the flow direction, and w denotes the width of the strip.

The above matrix equation can be written explicitly in the form

$$E_x = \rho_{xx}(B) J_x, \tag{15.33a}$$

$$E_y = \rho_{yx}(B) J_x. \tag{15.33b}$$

Thus the diagonal element $\rho_{xx}(B)$ of the magnetoresistivity tensor is measured by the ratio between the longitudinal electric field E_x and the current density J_x in the x-direction. The off-diagonal component $\rho_{yx}(B)$ is measured by the ratio between the transverse electric field E_y and the current density J_x. It can also be inferred by inspection that $\rho_{xy}(B) = -\rho_{yx}(B)$ (as shown also in the models discussed below).

The transverse electric field E_y, also called *Hall field*, is produced by the space charges accumulated (in stationary conditions) at the borders of the conductor because of the deflection due to the magnetic field. One or the other of the off-diagonal magnetoresistivity components (i.e. ρ_{yx} or its opposite ρ_{xy}) are also known as *Hall resistivity*. Most often it is convenient to report the *Hall coefficient*, defined as

$$R_{\text{Hall}}(B) = \frac{1}{B}\rho_{yx}(B) = \frac{1}{B}\frac{E_y}{J_x}. \tag{15.34a}$$

Notice that the *Hall potential* is $V_{\text{Hall}} = E_y w$, where w is the transverse dimension of the sample; in the absence of magnetic field, both V_{Hall} and ρ_{yx} vanish. The current in the Hall bar is related to the current density J_x, and to the thickness d and width w of the strip by the relation $I = J_x w d$. We have thus

$$\rho_{yx}(B) = \frac{E_y}{J_x} = \frac{V_{\text{Hall}}\, d}{I} \qquad \Longrightarrow \qquad V_{\text{Hall}} = \rho_{yx}(B)\frac{I}{d}; \tag{15.34b}$$

thus at parity of other conditions the Hall potential is higher for bars of small thickness. The opportunity to use conductors of small thickness, was at the basis of the discovery by Hall of the effect that brings his name, after failing in observing the effect in massive metal samples. *"Owing probably to the fact that the metal disk used had considerable thickness, the experiment at that time failed to give any positive result. Prof. Rowland now advised me, in repeating this experiment, to use gold leaf mounted on a plate of glass as my metal strip. I did so, and, experimenting as indicated above, succeeded*

on the 28th of October in obtaining, as the effect of the magnet's action, a decided deflection of the galvanometer needle …" [Excerpt from the article by E. H. Hall, Amer. J. Math. *2*, 287 (1879)].

We pass now to study the Hall effect and the magnetoresistivity in a few simple models. We consider first the case of a single type of carriers in a parabolic band model, with a unique relaxation time. Next, we consider the case in which holes and electrons are present, both with isotropic masses. The models described below, provide an indicative picture of the transport phenomena in the presence of magnetic fields in somewhat idealized situations. We wish to remark that the description of magneto-transport effects in realistic materials is rather demanding and requires a proper account of several features (such as energy dependence of the relaxation time, deviations from parabolic bands, detailed shape of the Fermi surfaces especially in the presence of a complicated connectivity in the repeated zone scheme, accurate analysis of the Boltzmann transport equations). We cannot enter in these and other aspects, and we refer for more elaborated models and discussions to the classic book by R. A. Smith "Semiconductors" (Cambridge University Press, Cambridge, 1978)].

Model 1. Magnetoresistivity and Hall Effect in an Isotropic One-Band Model

We consider here the magnetoresistivity and the Hall effect in the case of a single type of carriers (electrons or holes) in a parabolic energy band. For simplicity we use a model approach to the motion of electrons (or holes); the treatment with the more rigorous Boltzmann equation would give in the present case the same results.

The classical equation of motion of an electron, in a dissipative medium, in the presence of an electric field **E**, and a magnetic field **B** reads

$$m^* \frac{d\mathbf{v}}{dt} = (-e)\mathbf{E} + \frac{(-e)}{c}\mathbf{v} \times \mathbf{B} - \frac{m^*}{\tau}\mathbf{v}, \tag{15.35a}$$

where m^* is the effective mass of the electron, and a damping term with constant relaxation time τ has been included. In stationary conditions $d\mathbf{v}/dt = 0$, and Eq. (15.35a) becomes

$$\mathbf{v} = -\frac{e\tau}{m^*}\mathbf{E} - \frac{e\tau}{m^* c}\mathbf{v} \times \mathbf{B}. \tag{15.35b}$$

We specify the above equation in the geometry of Figure 15.9, with the electric field in the xy plane and the magnetic field in z-direction; Eq. (15.35b) becomes

$$\begin{cases} v_x = -\dfrac{e\tau}{m^*} E_x - \omega_c \tau v_y, \\ v_y = -\dfrac{e\tau}{m^*} E_y + \omega_c \tau v_x, \end{cases} \tag{15.36a}$$

where $\omega_c = eB/m^* c$ is the cyclotron frequency. From Eqs. (15.36a) we have

$$\begin{cases} v_x = -\dfrac{e\tau}{m^*} \dfrac{1}{1+\omega_c^2 \tau^2} (E_x - \omega_c \tau E_y), \\ v_y = -\dfrac{e\tau}{m^*} \dfrac{1}{1+\omega_c^2 \tau^2} (\omega_c \tau E_x + E_y). \end{cases} \tag{15.36b}$$

Thus the current density $\mathbf{J} = n(-e)\mathbf{v}$, (where n is the electron density) is related to the electric field via the magnetoconductivity tensor $\sigma(B)$ given by

$$\sigma(B) = \frac{ne^2\tau}{m^*}\frac{1}{1+\omega_c^2\tau^2}\begin{pmatrix} 1 & -\omega_c\tau \\ \omega_c\tau & 1 \end{pmatrix}. \tag{15.37}$$

Notice that $\sigma_{xy}(B) = -\sigma_{yx}(B)$, which is a particular case of the general Onsager relations.

Equation (15.37) provides the magnetoconductivity for a *parabolic band with constant (i.e. energy independent) relaxation time*. Inversion of the matrix (15.37) gives the magnetoresistivity tensor

$$\rho(B) = \frac{m^*}{ne^2\tau}\begin{pmatrix} 1 & +\omega_c\tau \\ -\omega_c\tau & 1 \end{pmatrix}. \tag{15.38}$$

From Eq. (15.38), we see that the diagonal (or parallel) magnetoresistivity $\rho_{xx}(B)$, the Hall magnetoresistivity $\rho_{yx}(B)$, and the Hall coefficient have the expressions

$$\rho_{xx}(B) = \frac{m^*}{ne^2\tau}, \quad \rho_{yx}(B) = -\frac{B}{nec}, \quad R_{\text{Hall}}(B) = -\frac{1}{nec}. \tag{15.39}$$

Thus in the parabolic one-band model with a single relaxation time, the diagonal magnetoresistivity turns out to be independent of B, and we have $\rho_{xx}(B) = \rho_{xx}(0) = m^*/(ne^2\tau)$. Even more important, the Hall coefficient is independent of the effective mass and of the relaxation time; it depends only on the carrier concentration and charge sign. Also notice that in the case of positive holes, the off-diagonal matrix elements in Eqs. (15.37)–(15.39) change sign.

The results summarized in Eq. (15.39), obtained in the rather idealized one-band model, are to be taken only as indicative, and cannot be used as they stand for quantitative descriptions of realistic conductors. It is important in fact to notice that a proper account of the energy dependence of the relaxation time, or of the anisotropy of the energy bands, modify the results of Eq. (15.39); in particular, a dependence of $\rho_{xx}(B)$ on B is actually always observed in experiments. For these reasons, we consider the slightly more sophisticated two-band model, representing two groups of carriers.

Model 2. Magnetoresistivity and Hall Effect in an Isotropic Two-Band Model

Interesting new features appear in the study of magnetoresistivity and Hall effect within the two-band model. For simplicity we suppose that the two bands are parabolic, with effective masses m_1 and m_2; we also assume that the relaxation times τ_1 and τ_2 are constant for each group of carriers. The two-band model is useful to provide insight of transport phenomena in crystals with two groups of carriers of the same type (but different masses or relaxation times) or for mixed type carriers; we consider specifically this last situation.

Consider a material with n electrons (per unit volume) of mass m_1 and relaxation time τ_1, and p holes of mass m_2 and relaxation time τ_2. The magnetoconductivity is

just the sum of the contributions from each group of carriers. Using Eq. (15.37) for electrons, and the appropriate modified form for positive holes, we obtain

$$\sigma(B) = \begin{pmatrix} A_1 & -B_1 \\ B_1 & A_1 \end{pmatrix} + \begin{pmatrix} A_2 & B_2 \\ -B_2 & A_2 \end{pmatrix} = \begin{pmatrix} A_1 + A_2 & -B_1 + B_2 \\ B_1 - B_2 & A_1 + A_2 \end{pmatrix}, \tag{15.40}$$

where

$$A_1 = \frac{\sigma_1}{1 + \omega_1^2 \tau_1^2}, \quad B_1 = \frac{\sigma_1 \omega_1 \tau_1}{1 + \omega_1^2 \tau_1^2}, \quad \sigma_1 = \frac{n e^2 \tau_1}{m_1}, \tag{15.41a}$$

$$A_2 = \frac{\sigma_2}{1 + \omega_2^2 \tau_2^2}, \quad B_2 = \frac{\sigma_2 \omega_2 \tau_2}{1 + \omega_2^2 \tau_2^2}, \quad \sigma_2 = \frac{p e^2 \tau_2}{m_2}. \tag{15.41b}$$

The magnetoresistivity tensor is obtained by inverting the magnetoconductivity tensor (15.40); it holds

$$\rho(B) = \frac{1}{(A_1 + A_2)^2 + (B_1 - B_2)^2} \begin{pmatrix} A_1 + A_2 & B_1 - B_2 \\ -B_1 + B_2 & A_1 + A_2 \end{pmatrix}. \tag{15.42}$$

Consider first the parallel component $\rho_{xx}(B)$ of the magnetoresistivity tensor of Eq. (15.42); using expressions (15.41) one obtains

$$\rho_{xx}(B) = \frac{\sigma_1 + \sigma_2 + \sigma_1 \omega_2^2 \tau_2^2 + \sigma_2 \omega_1^2 \tau_1^2}{(\sigma_1 + \sigma_2)^2 + (\sigma_1 \omega_2 \tau_2 - \sigma_2 \omega_1 \tau_1)^2}. \tag{15.43}$$

It is straightforward to verify that $\rho_{xx}(B) > \rho_{xx}(0)$; thus $\rho_{xx}(B) - \rho_{xx}(0)$ is an essentially *positive quantity for any value of the magnetic field*.

For very high magnetic fields (such that $\omega_1 \tau_1 \gg 1$ and $\omega_2 \tau_2 \gg 1$), Eq. (15.43) shows that in general $\rho_{xx}(B \to \infty)$ is finite, and thus there is saturation of the magnetoresistivity; the only remarkable exception occurs when

$$\sigma_1 \omega_2 \tau_2 \equiv \sigma_2 \omega_1 \tau_1 \qquad \Longrightarrow \qquad n = p.$$

When the two groups of carriers of opposite type (electrons and holes) have the same concentration, then $\rho_{xx}(B \to \infty) = \infty$ and no saturation occurs.

Consider now the off-diagonal magnetoresistivity transport parameter $\rho_{yx}(B)$; from Eqs. (15.42) and (15.41) one obtains

$$\rho_{yx}(B) = \frac{-\sigma_1 \omega_1 \tau_1 (1 + \omega_2^2 \tau_2^2) + \sigma_2 \omega_2 \tau_2 (1 + \omega_1^2 \tau_1^2)}{(\sigma_1 + \sigma_2)^2 + (\sigma_1 \omega_2 \tau_2 - \sigma_2 \omega_1 \tau_1)^2}. \tag{15.44a}$$

For high magnetic fields (i.e. $\omega_i \tau_i \gg 1$), the above expression simplifies in the form

$$\rho_{yx}(B \to \infty) \approx -\frac{\omega_1 \tau_1 \omega_2 \tau_2}{\sigma_1 \omega_2 \tau_2 - \sigma_2 \omega_1 \tau_1} = -\frac{B}{(n - p) e c}. \tag{15.44b}$$

The Hall parameter becomes

$$R_{\text{Hall}}(B \to \infty) = -\frac{1}{(n - p) e c}, \tag{15.44c}$$

a result which is independent of relaxation time and is *governed by the difference of the density of electrons and holes*. It is easy to understand qualitatively the limiting result (15.44c); for high values of B, the deflection of carriers produced by the magnetic field increases. Since electrons and holes have opposite charges and move in opposite directions, they are deflected on *the same side;* thus the effective number of carriers entering in Eq. (15.44c) is given by the difference of the electron and hole concentrations.

15.6 Quantum Hall Effects

15.6.1 Integer Quantum Hall Effect

In the previous section we have considered some effects of magnetic fields on transport properties in three-dimensional materials. In this section we present some aspects of transport measurements under strong magnetic fields for the two-dimensional electron gas; the observed effects have opened new areas of investigation and brought major breakthroughs in the comprehension of the two-dimensional structures and new states of condensed matter.

The transport properties of *two-dimensional* conductors, when observed in high purity samples, at very low temperatures and strong magnetic fields, show striking departure from the classical behavior; in particular the Hall resistance $\rho_{xy}(B)$ versus B exhibits flat plateaus, from which one can obtain the universal constant h/e^2, now known as the von Klitzing constant. In the plateau regions the Hall resistance is given exactly by h/e^2 divided by an integer, and its experimental value is

$$R_K = \frac{h}{e^2} = 25812.807\ \Omega. \tag{15.45}$$

Measurements, performed even on samples of different origin, turn out to be reproducible within the astonishing accuracy of one part per billion (or so), and for this reason the Hall effect has attained a special role in metrology as the standard of resistance [see for instance P. J. Mohr, B. N. Taylor and D. B. Newell "CODATA recommended values of the fundamental physical constants: 2006" Rev. Mod. Phys. *80*, 633 (2008)].

The quantum Hall effect was first reported for the two-dimensional electron gas in the inversion layer of a silicon metal-oxide-semiconductor field-effect-transistor at $T = 1.5$ K and $B = 18$ Tesla by K. von Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. *45*, 494 (1980). In the device, the density of surface electrons can be controlled and changed by varying the MOSFET gate voltage; the Hall resistance shows fixed values $(1/i)(h/e^2)$ (with i integer number) around experimentally well-defined surface carriers concentrations, while the longitudinal resistance is vanishingly small.

Degenerate two-dimensional electron systems can be also realized at the interface between GaAs and (n-doped) $Al_xGa_{1-x}As$; nearly ideal semiconductor heterostructures are prepared by molecular beam epitaxy techniques. As already discussed in Section 14.4, the electrons at the interface are confined by the potential well originated from the conduction band offset; the motion perpendicular to the interface is quantized, and, even when all the carriers are trapped in the lowest ground state, the motion