2.6.3
The magneto-resistivity tensor

We proceed by studying the electron transport according to Eq. (2.57) in weak magnetic fields, with "weak" being specified by

$$\omega_c = e|\vec{B}|/m_e^* \ll 1/\tau \tag{2.58}$$

This condition means that the distance the electrons travel before getting scattered (the *mean free path* $\ell_e \equiv v_F \tau$) is small compared to the cyclotron circumference $2\pi r_c$. We will see in Chapters 6 and 7 what happens when the electrons can complete the cyclotron orbits without getting scattered. Suppose a magnetic field is applied in the z-direction, $\vec{B} = (0, 0, B)$. In such a case, we obtain

$$j_x = \sigma E_x + \sigma v_y B = \sigma E_x + \frac{ne^2\tau}{m_e^*} v_y B = \sigma E_x - j_y \omega_c \tau$$

$$j_y = \sigma E_y - \sigma v_x B = \sigma E_y - \frac{ne^2\tau}{m_e^*} v_x B = \sigma E_y + j_x \omega_c \tau$$

$$j_z = \sigma E_z$$

where v_i are the components of the drift velocity vector. Solving this system of equations for $\vec{j}$ gives $\vec{j} = \underline{\sigma} \vec{E}$ with

$$\underline{\sigma} = \frac{\sigma}{1 + \omega_c^2 \tau^2} \begin{pmatrix} 1 & -\omega_c \tau & 0 \\ \omega_c \tau & 1 & 0 \\ 0 & 0 & 1 + \omega_c^2 \tau^2 \end{pmatrix} \tag{2.59}$$

Here $\underline{\sigma}$ is known as the magneto-conductivity tensor. Its components can be experimentally determined by measuring four-probe resistances using "Hall bar" shaped samples (Fig. 2.13). Voltage probes are attached to a rectangular thin film of the material, aligned parallel to the x- and y-directions, and perpendicular to the magnetic field direction. The transport in the z-direction remains unaffected by $\vec{B}$ and is of no further interest to us. We can determine the components ρ_{xx} and ρ_{xy} of the resistivity tensor by applying a current in the x-direction and measuring the voltage drops V_x and V_y. Since
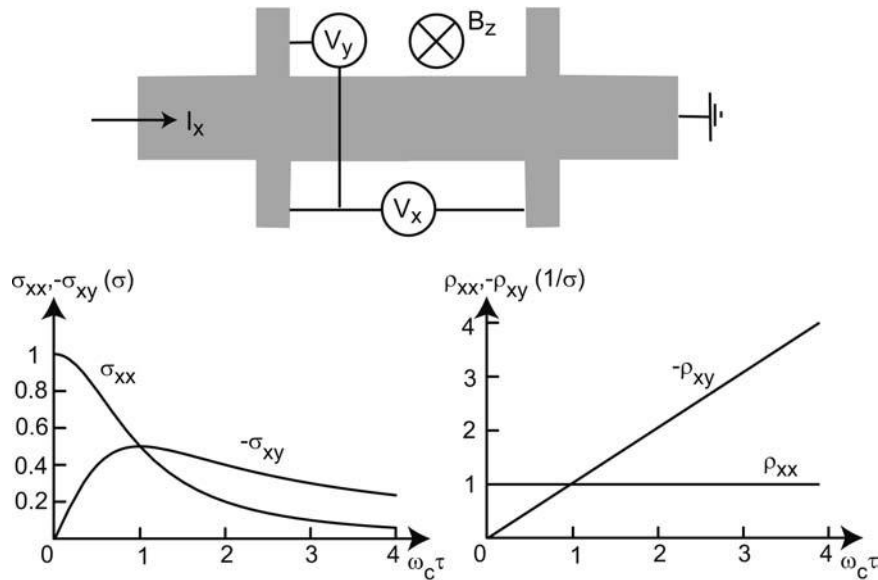
$$\begin{pmatrix} V_x \\ V_y \end{pmatrix} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ -\rho_{xy} & \rho_{xx} \end{pmatrix} \cdot \begin{pmatrix} I_x \\ I_y \end{pmatrix} \cdot S$$

and

$$\begin{pmatrix} I_x \\ I_y \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{xx} \end{pmatrix} \cdot \begin{pmatrix} V_x \\ V_y \end{pmatrix} \cdot \frac{1}{S}$$

where S is a geometry factor (see [235]), we can establish the relation between the components of the resistivity and the conductivity tensors:

$$\rho_{xx} = \frac{\sigma_{xx}}{\sigma_{xx}^2 + \sigma_{xy}^2}, \qquad \rho_{xy} = \frac{-\sigma_{xy}}{\sigma_{xx}^2 + \sigma_{xy}^2} \tag{2.60}$$

Fig. 2.13 Top: Top view of a Hall geometry. The magnetic field is applied perpendicular to the sheet. Bottom: The components of the conductivity and the resistivity tensors are shown to the left and to the right, respectively.

We thus find that ρ_{xx} does not depend on $\vec{B}$, and $\rho_{xy} = -B/en = R_{\mathrm{H}}B$. Here ρ_{xy} is the Hall resistivity, and $R_{\mathrm{H}} = -1/en$ is known as the Hall coefficient. Hall measurements are actually a standard tool to determine carrier densities. It may be counter-intuitive at first sight that, for $\rho_{xx} = 0$, σ_{xx} becomes zero as well. Furthermore, the Onsager–Casimir symmetry relation should be mentioned, which states that the result of a measurement is exactly the same when all current and voltage sources are exchanged, and the polarity of the magnetic field is reversed. One consequence is that two-probe measurements, in which the voltage drop is measured between the source and drain contacts, must be symmetric with respect to $B = 0$.

Question 2.8: *Write down Eqs. (2.60) for an anisotropic sample.*

2.6.4
Diffusion currents

In close analogy to the treatment of drift currents, the Boltzmann model can be applied to diffusion currents, i.e. currents as a consequence of a position-dependent varying chemical potential $\mu(\vec{r})$, which can have its origin in a gradient of the temperature or of the carrier density. Assuming constant tem-

perature and no external electric fields, the Boltzmann treatment results in a diffusion current density

$$\vec{j}_{\text{diff}} = \frac{n\sigma_0}{e}\vec{\nabla}\mu(\vec{r}) \tag{2.61}$$

The diffusion current is frequently expressed in terms of the carrier density gradient and the diffusion constant D,

$$\vec{j}_{\text{diff}} = eD\vec{\nabla}n(\vec{r}) \tag{2.62}$$

Question 2.9: *Show that for a quasi-free electron gas, the relation between D and the mobility μ equals*

$$D = \frac{2E_{\text{F}}}{3e}\mu \qquad\qquad \text{in } d = 3$$

$$D = \frac{E_{\text{F}}}{e}\mu \qquad\qquad \text{in } d = 2$$

$$D = \frac{2E_{\text{F}}}{e}\mu \qquad\qquad \text{in } d = 1 \tag{2.63}$$

Such relations are known as Einstein relations.[4]

2.7
Scattering mechanisms

As mentioned in Section 2.6, many scattering mechanisms contribute to the average momentum relaxation time τ. Each process has its characteristic matrix element $W_{\vec{k},\vec{k}'}$, Eq. (2.49). The relevance of a particular kind of scattering varies greatly and depends on the carrier density as well as on the temperature. How in detail the matrix elements are calculated is treated in several excellent books, e.g. [254, 270]. Each scattering mechanism can be characterized by its contribution to the carrier mobility μ_i, which sum up to the total mobility according to the Matthiesen rule, $1/\mu = \sum_i 1/\mu_i$. In pure crystals, the sole source of scattering is lattice vibrations. Electron–phonon scattering has several facets. In crystals with valley degeneracy, electrons may be scattered between valleys, which requires absorption or emission of a phonon. In polar and/or piezoelectric crystals, on the other hand, lattice vibrations go along with strong oscillating electric fields. In real crystals, charged impurities may dominate the scattering rates. We briefly present the most important scattering mechanisms below.

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4) The Einstein relation for particles obeying the Boltzmann statistics is Eq. (5.10).

An impurity breaks the symmetry of the lattice and causes scattering. If the impurity is neutral, the scattering rates are usually negligible. Charged impurities, however, represent screened Coulomb scatterers, with peak potentials that can become comparable to the Fermi energy.[5] Clearly, an electron with a larger kinetic energy will get deflected by a smaller angle as it gets scattered, and we can expect that the mobility increases as the temperature, and with it the average electron kinetic energy, increases. In fact, an evaluation of the corresponding matrix element shows that, for weak Coulomb potentials and within the Born approximation, the resulting mobility is $\propto \Theta^{3/2}$, multiplied by a logarithmic correction, i.e. a factor that depends logarithmically on Θ.

Electron–phonon scattering can be divided into deformation potential scattering and scattering of electrons by the corresponding electric fields. By deformation potential scattering, we mean scattering at the lattice deformations caused by the phonons. Here, scattering at acoustic phonons is the most important mechanism. Since the energy transfers are small in electron–acoustic phonon scattering, it can be treated as quasi-elastic. A simple argument gives the correct temperature dependence. The density of acoustic phonons n_{ac} is proportional to the Bose–Einstein distribution, which, for large temperatures compared to the phonon energy, varies as $1/\Theta$. Since the mobility is proportional to $n_{\mathrm{ac}}/\bar{v}$ ($\bar{v}$ is the average electron velocity, which is $\propto \sqrt{\Theta}$), we expect that the mobility due to electron–acoustic phonon scattering is $\propto \Theta^{-3/2}$. This is in fact observed experimentally.

Furthermore, both optical and acoustic phonons can assist the electron in scattering between the valleys in a crystal with valley degeneracy, such as Si. The corresponding momentum transfers are quite large, since the separation of the valley in reciprocal space is of the order of the size of the Brillouin zone.

This completes the list of the scattering mechanisms relevant in Si. In this material, ionized impurities dominate the mobility at low temperatures, while quasi-elastic acoustic phonon scattering is the most important mechanism at intermediate temperatures. For $\Theta > 200\,\mathrm{K}$, inter-valley scattering becomes significant as well. Consequently, the mobility in Si shows a maximum as a function of temperature. Its position depends on both the impurity density and the carrier density. Electron mobilities up to $1\,\mathrm{m}^2/\mathrm{V}\,\mathrm{s}$ have been achieved in Si.

GaAs is a polar material, and consequently lattice vibrations are always accompanied by oscillating electric fields. They are particularly strong for optical phonons. The resulting scattering mechanism is called polar scattering. Optical phonons vanish for temperatures below $\approx 60\,\mathrm{K}$, and consequently polar scattering is relevant only above this temperature. In the limit $k_{\mathrm{B}}\Theta \gg \hbar\omega_{\mathrm{op}}$ (ω_{op} denotes the optical phonon frequency, which for GaAs is of the order of $5\,\mathrm{meV}$; see Fig. 2.14), it can be shown that the resulting mobility varies as

5) The screened Coulomb potential is studied in Exercise E2.5.
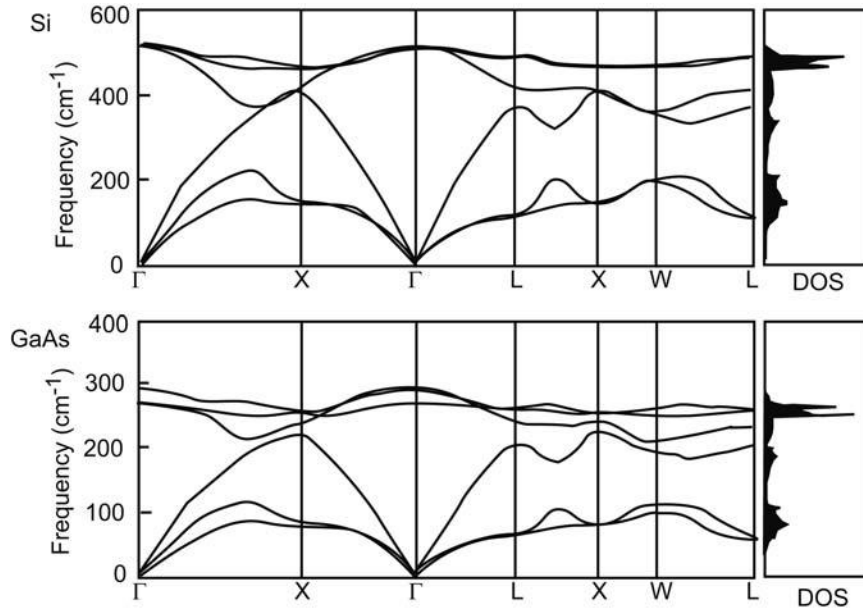
Fig. 2.14 Phonon dispersions for Si (top) and GaAs (bottom). After [118].

$\Theta^{-1/2}$. If the crystal is piezoelectric like GaAs, a crystal deformation generates a polarization field as well, which is another source of scattering, called piezoelectric scattering. As for the polar scattering, the mobility due to piezoelectric scattering is $\propto \Theta^{-1/2}$, although this temperature dependence holds for a larger range of temperatures.

Fig. 2.15 summarizes the contributions of different scattering mechanisms to the electron total mobility of GaAs. A comparison with measurements reveals that, at low temperatures, ionized impurity scattering dominates, while, at higher temperatures, the mobility is entirely determined by polar scattering. In a small temperature range around the emerging maximum of the mobility, piezoelectric scattering is significant. Furthermore, it is seen that acoustic phonon scattering plays no role, in contrast to the scattering in Si.

2.8
Screening

The conduction electrons react to perturbations. They collect in the potential valleys and avoid the peaks. As a consequence, the external potential is reduced to an effective potential in the crystal; the electrons "screen" the perturbation. The goal of this section is to present a qualitative picture of how