We consider the electrons to be non-interacting. Strict justification of this comes from the Landau theory, in which an interacting electron gas is re-described in terms of non-interacting *quasi-particles* with renormalised energy (as compared to original particles) and a finite lifetime. Providing this lifetime is long compared to any experimentally relevant processes, the quasi-particle picture is a valid one, and this is generally the case in semiconductors. We may also appeal to *a posteriori* justification, as we will see that this simple treatment is sufficient to describe a wide range of interesting mesoscopic transport experiments.

3.3 Effects of confinement

In the triangular quantum well of Fig. 3.1, confinement in one spatial dimension is much stronger than in the other two. With z singled out as the strongly confined dimension, we may therefore approximate the confinement potential as $U(\mathbf{r}) = U(z)U(x,y)$. With magnetic field in the z direction, i.e. perpendicular to the plane of the interface, the effective mass Schrödinger Equation, Eq. (3.1), admits the separable solution $\Psi(\mathbf{r}) = \phi_n(z)\psi(x,y)$, with $\phi_n(z)$ the nth quantised solution of the one-dimensional problem in the z direction. Index $n = 1, 2, \ldots$ defines a set of *sub-bands*; if we consider the electrons to be unconfined in the plane of the interface, then the full eigenfunctions of Eq. (3.1) with $U(\mathbf{r}) = U(z)$ are

$$\Psi(\mathbf{r}) = \phi_n(z) e^{ik_x x} e^{ik_y y} \tag{3.2}$$

with dispersion

$$E(n,k) = E_c + \epsilon_n + \frac{\hbar^2}{2m^*}\left(k_x^2 + k_y^2\right) \tag{3.3}$$

with ϵ_n, the eigen-energies from z-confinement.

The density of states (per unit energy, per unit surface area) of such a quasi-infinite two-dimensional system is

$$\mathcal{D}(E) = \sum_n \frac{m^*}{\pi\hbar^2}\theta\left(E - \epsilon_n - E_c\right) = \sum_n \mathcal{D}_0 \theta\left(E - \epsilon_n - E_c\right), \tag{3.4}$$

with $\theta(E)$ the unit step function and where a factor 2 for spin has been included. Within the nth subband then, the density of states is constant, with value $n\mathcal{D}_0$. For GaAs, with effective mass $m^* = 0.07 m_e$, $\mathcal{D}_0 = 2.9 \times 10^{10}/(\text{cm.meV})$.

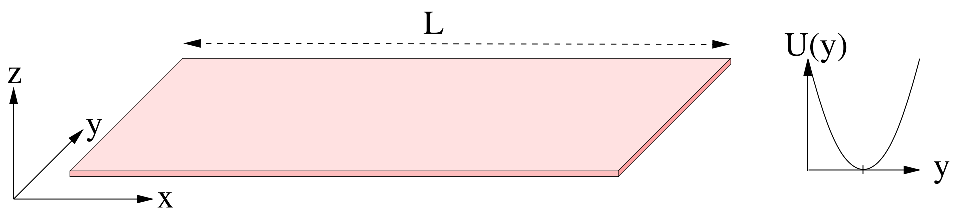

Figure 3.3: Sketch of a 2DEG, establishing co-ordinate system. Strong confinement is in the z direction, and much weaker parabolic confinement in the y direction. The extent of the sample in the x-direction is large cf. extent in y and z directions.

Confinement in the z direction is strong enough that experiments are usually restricted to the lowest $n = 1$ sub-band. In both 2DEGs and SAQD the z-confinement is $\sim$ 10 times that in the x-y plane. Thus, sub-bands $n \geq 2$ play no significant role and we can neglect the z direction altogether — reducing the original 3D problem to a two-dimensional one with effective 2D Schrödinger equation

$$\left[E_s + \frac{1}{2m^*}\left(i\hbar\boldsymbol{\nabla} + e\mathbf{A}\right)^2 + U(x,y)\right]\psi(x,y) = E\psi(x,y) \tag{3.5}$$

with $E_s = E_c + \epsilon_1$, and 2D vector operators.

3.4 Transverse modes in 2DEG

Consider a uniform 2D conductor, much longer than it is wide (Fig. 3.3). We will consider transport parallel to the long axis of the conductor (x direction), assuming that the motion is essentially unconfined in this direction. In the transverse (y) direction, we model the confinement with a harmonic potential, such that we write

$$U(x,y) = U(y) = \frac{1}{2}m^*\omega_0^2 y^2, \tag{3.6}$$

with ω_0 the *confinement energy* in the y direction. Harmonic confinement is a convenient mathematical form as it leads to analytical solutions. It also provides a reasonable approximations to confinements found in experiment.

We consider an applied magnetic field perpendicular to the 2DEG (in the z direction), and choose a gauge such that the vector potential is written

$$\mathbf{A} = -\hat{\boldsymbol{e}}_x By; \qquad A_x = -By; \qquad A_y = 0. \tag{3.7}$$

The 2D Schrödinger equation, Eq. (3.5), can then be written

$$\left[E_s + \frac{1}{2m^*}(p_x + eBy)^2 + \frac{1}{2m^*}p_y^2 + U(y)\right]\psi(x,y) = E\psi(x,y), \qquad (3.8)$$

with $p_x = -i\hbar\partial/\partial x$ and $p_y = -i\hbar\partial/\partial y$. This has solution

$$\psi(x,y) = \frac{1}{\sqrt{L}}e^{ikx}\chi(y) \qquad (3.9)$$

with plane wave in x direction (normalised to length L) and the transverse function $\chi(y)$ given by the solution of the 1D problem

$$\left[E_s + \frac{1}{2m^*}p_y^2 + \frac{1}{2m^*}(\hbar k + eBy)^2 + \frac{1}{2}m^*\omega_0^2 y^2\right]\chi(y) = E\chi(y). \qquad (3.10)$$

Let us define the cyclotron frequency

$$\omega_c = \frac{|eB|}{m^*}, \qquad (3.11)$$

and the length

$$y_k = \frac{\hbar k}{eB}. \qquad (3.12)$$

We have then

$$\left[E_s + \frac{1}{2m^*}p_y^2 + \frac{1}{2}m^*\omega_c^2(y + y_k)^2 + \frac{1}{2}m^*\omega_0^2 y^2\right]\chi(y) = E\chi(y). \qquad (3.13)$$

Completing the square, we have

$$\left[E_s + \frac{m^*}{2}\frac{\omega_0^2\omega_c^2}{\widetilde{\omega}^2}y_k^2 + \frac{1}{2m^*}p_y^2 + \frac{1}{2}m^*\widetilde{\omega}^2\left(y + \frac{\omega_c^2}{\widetilde{\omega}^2}y_k\right)^2\right]\chi(y) = E\chi(y), \qquad (3.14)$$

with

$$\widetilde{\omega}^2 = \omega_c^2 + \omega_0^2 \qquad (3.15)$$

This, then, has the form of a displaced Harmonic oscillator, and from elementary quantum mechanics, we have the solutions

$$\chi_{n,k}(y) = u_n\left(q + \frac{\omega_c^2}{\widetilde{\omega}^2}q_k\right), \qquad (3.16)$$

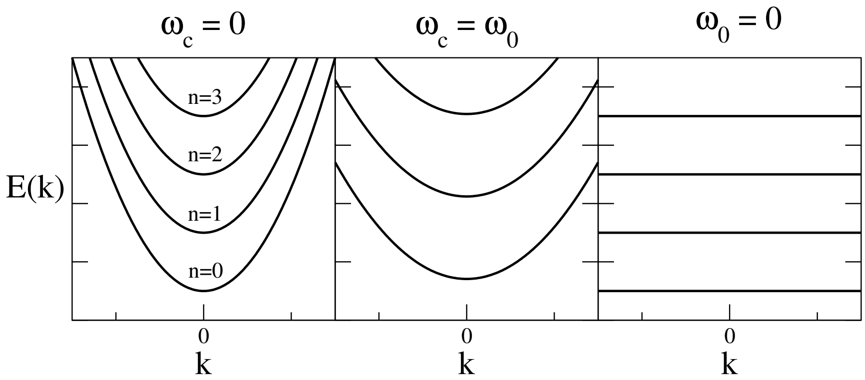

Figure 3.4: Dispersion relation for a 2DEG with transverse harmonic confinement and perpendicular magnetic field. The three plots are for different choices of ω_0 and ω_c, the confinement and cyclotron frequencies, respectively.

with $u_n(x)$ the usual simple-harmonic oscillator eigenfunctions written in terms of the dimensionless displacements $q = y/\widetilde{l}$ and $q_k = y_k/\widetilde{l}$ with length

$$\widetilde{l} = \sqrt{\frac{\hbar}{m^* \widetilde{\omega}}}. \tag{3.17}$$

The corresponding dispersion relation is

$$\begin{aligned} E &= E_s + \frac{m^*}{2} \frac{\omega_0^2 \omega_c^2}{\widetilde{\omega}^2} y_k^2 + \left(n + \frac{1}{2}\right) \hbar\widetilde{\omega} \\ &= E_s + \left(n + \frac{1}{2}\right) \hbar\widetilde{\omega} + \frac{\hbar^2 k^2}{2m^*} \frac{\omega_0^2}{\widetilde{\omega}^2}, \end{aligned} \tag{3.18}$$

with $n = 0, 1, 2, \ldots$. This result is illustrated in Fig. 3.4

The first thing to notice is that due to the confinement in the y-direction, we obtain a series of sub-bands, labelled with quantum number n. In contrast to the z-direction, however, here the confining potential is relatively weak, and more than just the lowest of sub-band will play a role in transport. In analogy with optical wave guides, these sub bands are known as *transverse modes* and they play a crucial role in determining the transport properties of low-dimensional conductors. We also note that the dispersion of a given transverse mode is of plane-wave form (i.e. quadratic) but with a renormalised mass $m^* \to m^* \left(1 + \omega_c^2/\omega_0^2\right)$ — increasing the magnetic field increases this renormalised mass of the electrons and makes the dispersion relation flatter. Figure 3.4 highlights two limiting cases:

- **Zero field:** For $B \to 0$, we have $\omega_c \to 0$ and $\widetilde{\omega} \to \omega_0$ such that

$$E = E_s + \left(n + \frac{1}{2}\right)\hbar\omega_0 + \frac{\hbar^2 k^2}{2m^*}, \tag{3.19}$$

  as expected.

- **Zero confinement** For $\omega_0 \to 0$, we have $\widetilde{\omega} \to \omega_c$ and

$$E = E_s + \left(n + \frac{1}{2}\right)\hbar\omega_c, \tag{3.20}$$

  in which case, quantum number n therefore the familiar Landau levels with quantisation energy given by the cyclotron frequency. NB: there is no dispersion in this limit.

3.5 Quantum dots: Fock-Darwin Spectrum

A useful model for the electronic confinement of a quantum dot is the two-dimensional, symmetric parabolic potential

$$U(x, y) = \frac{m^*}{2}\omega_0^2\left(x^2 + y^2\right). \tag{3.21}$$

The corresponding Hamiltonian of an electron in the dot is

$$H = \frac{1}{2m^*}\left(\mathbf{p} + e\mathbf{A}\right)^2 + \frac{m^*}{2}\omega_0^2\left(x^2 + y^2\right). \tag{3.22}$$

By using the symmetric gauge for the vector potential $\mathbf{A} = (-By/2, Bx/2, 0)$ it can be shown (homework!) that the energy spectrum the dot is given by

$$E_{n_+,n_-} = (n_+ + 1)\,\hbar\Omega + \frac{1}{2}\hbar\omega_c n_-, \tag{3.23}$$

with

$$\Omega^2 = \omega_0^2 + \frac{\omega_c^2}{4}, \tag{3.24}$$

and quantum numbers

$$n_\pm = n_x \pm n_y; \quad \text{for} \quad n_x, n_y = 0, 1, 2, \ldots. \tag{3.25}$$

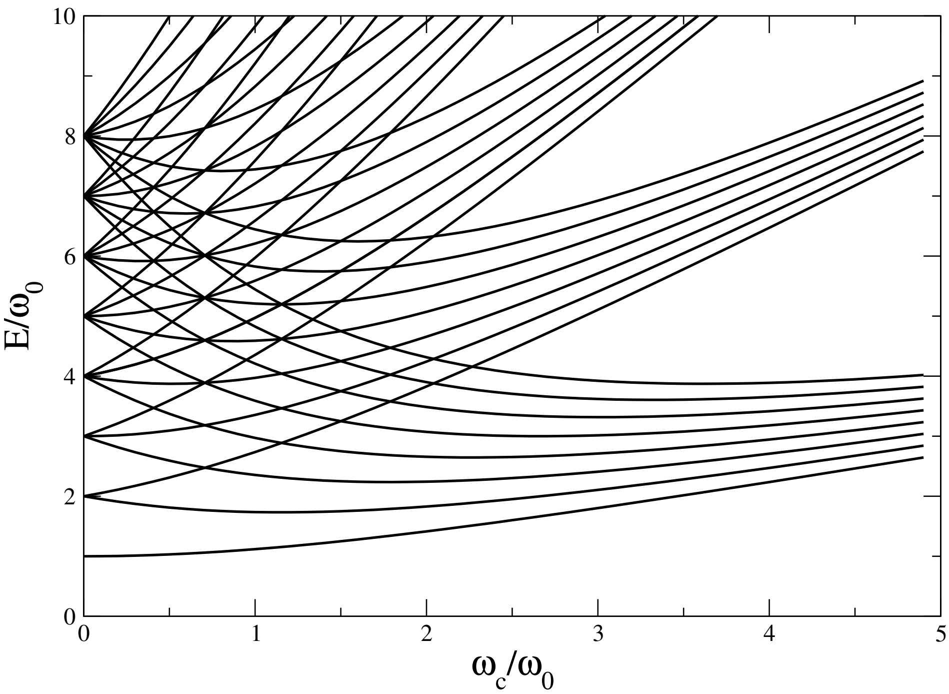

Figure 3.5: Fock-Darwin spectrum of symmetric quantum dot up to quantum number $n = 7$.

This result is known as the Fock-Darwin spectrum after the physicists who initially discussed the problem in the 1930s (nothing to do with quantum dots). This spectrum is plotted in Fig. 3.5. For $B = 0$ we have the regularly-spaced spectrum of a two-dimensional symmetric harmonic oscillator. In the high-field limit, the spectrum goes over into that of the Landau levels with the effects of the dot confinement playing an ever decreasing role.