These measurements have confirmed in a beautiful way the model of contact resistances and ballistic transport in one-dimensional systems. There is actually a quite different kind of experiment which proves this as well, namely four-terminal resistance measurements on 2DEGs in the quantum Hall regime. This is the topic of the following sections.

7.3 The Landauer–Büttiker formalism

In the previous section, we have argued that the quantized conductance of ballistic quantum wires stems from contact resistances. We have also seen that four-probe measurements give a resistance of zero, as expected from their interpretation in terms of contact resistances. In fact, a conceptually very similar system is a 2DEG in the quantum Hall regime. As already indicated in Section 6.2, the electrons skip along the edge of the Hall bar in strong magnetic fields. The origin of this dynamics is illustrated in Fig. 7.17.

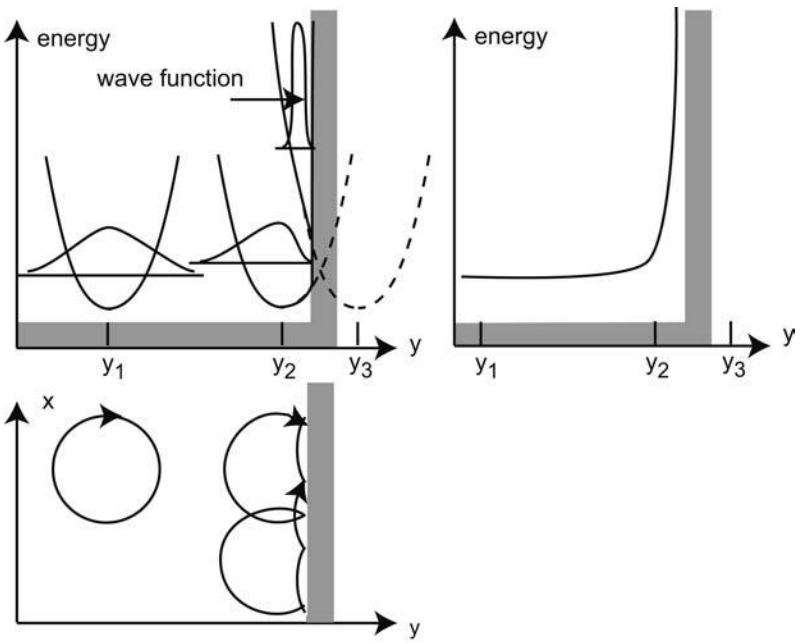

Fig. 7.17 Modification of the magnetoelectric confinement of the electrons as they approach the edge of the 2DEG (top left). The undisturbed cyclotron motion at y_1 is increasingly squeezed as the guiding center approaches the edge (positions y_2 and y_3). As a consequence, the energy of the Landau level increases (right), while the electrons delocalize along the x-direction (bottom).

7.3.1 Edge states

At the edge of the 2DEG, the conduction band bottom increases sharply and modifies the combined potential of the Landau harmonic oscillators and the electrostatic confinement. The increased confinement shifts the Landau levels to higher energies. Each LL crosses the Fermi level at some point, and consequently the density of states at the Fermi level is always larger than zero. As sketched in Fig. 7.17 as well, the electrons skip along the edge. Therefore, we speak of *skipping orbits* and *edge states*. Edge states have several peculiar features, which become self-evident immediately. First of all, they are one-dimensional: the electron motion is confined perpendicular to the sample edge, but is free in the direction parallel to it. Second, all the electrons at one sample edge move in the same direction, while the electrons at the opposite edge move in the opposite direction. In the bulk, all electrons are localized at potential modulations, except for special filling factors, as already shown in Section 6.2. The resulting edge state configuration with the directions of current flow is shown in Fig. 7.18.[6]

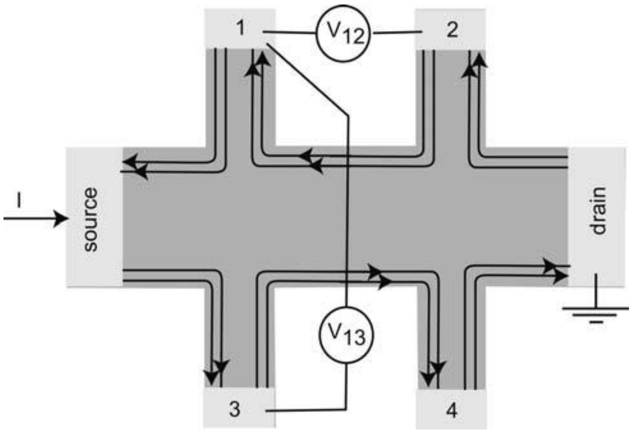

Fig. 7.18 Top view of a Hall bar in a strong magnetic field. Current flows in one-dimensional edge states only, in the directions indicated by the arrows. Here, two Landau levels are occupied.

There is no backscattering in edge states, i.e. the elastic mean free path approaches infinity. Suppose an electron in an edge state hits a scatterer close to the edge. Its momentum right after the scattering event may be reversed, but the strong magnetic field bends the momentum back into the forward direction. In order to be backscattered, the electron has to traverse the whole Hall bar and reach the opposite edge! Hence, backscattering is greatly reduced. It

6) Since the electrons circulate around the edge, one speaks of a *chiral* Fermi liquid.

follows that a 2DEG in the quantum Hall regime comes very close to an ideal ballistic quantum wire: it is one-dimensional and backscattering is absent. We can even attach voltage probes inside the quantum wires without inducing backscattering. Therefore, the voltage drop between, for example, contacts 1 and 2 in Fig. 7.18 should be zero. You will have realized, of course, that this is exactly what we measure in a Shubnikov–de Haas experiment. In [47], the Landauer formula has been generalized to an arbitrary number of contacts, such that circuits of ballistic quantum wires can be treated. The concept is known as the Landauer–Büttiker formalism.

Consider a circuit of ballistic quantum wires, like, the system of Fig. 7.18. We define the *direct* transmission probability of contact p into contact q as $T_{q \leftarrow p} = T_{qp}$. It is possible to have $T_{qp} > 1$, since more than one mode may connect the two contacts. Note that T_{qp} does not have to be an integer. Note further that, within this notation, T_{pp} is a backscattering probability. The *total current emitted by contact p* is denoted by I_p, while μ_p is the electrochemical potential of contact p. Again, an "ideal" contact absorbs all incoming electrons and distributes the emitted electrons equally among all outgoing modes, such that they are filled up to μ_p, assuming zero temperature.

In this notation, the Landauer formula generalizes to the Büttiker formula

$$I_p = \frac{2e}{h} \sum_q (T_{qp}\mu_p - T_{pq}\mu_q) \tag{7.34}$$

which is a direct consequence of current conservation. We proceed by applying the Büttiker formula to the sample shown in Fig. 7.18. It gives a system of six linearly dependent equations, one for each contact:

$$\begin{pmatrix} I_s \\ I_d \\ I_1 \\ I_2 \\ I_3 \\ I_4 \end{pmatrix} = \begin{pmatrix} G & 0 & -G & 0 & 0 & 0 \\ 0 & G & 0 & 0 & 0 & -G \\ 0 & 0 & G & -G & 0 & 0 \\ 0 & -G & 0 & G & 0 & 0 \\ -G & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & -G & G \end{pmatrix} \begin{pmatrix} V_s \\ V_d \\ V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix}$$

with $G = j(2e^2/h)$. By choosing $\mu_d = 0$ as a reference potential, and after eliminating the drain current as a consequence of current conservation (remember that the voltage probes measure the potentials without drawing current), we can eliminate the drain row and column, and the following matrix equation results:

$$\begin{pmatrix} I_s \\ I_1 \\ I_2 \\ I_3 \\ I_4 \end{pmatrix} = \begin{pmatrix} G & -G & 0 & 0 & 0 \\ 0 & G & -G & 0 & 0 \\ 0 & 0 & G & 0 & 0 \\ -G & 0 & 0 & G & 0 \\ 0 & 0 & 0 & -G & G \end{pmatrix} \begin{pmatrix} V_s \\ V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix}$$

Its solution gives

$$V_s = V_3 = V_4, \qquad V_1 = V_2 = 0, \qquad I_s = GV_s \tag{7.35}$$

a result that you may have guessed, considering, for example, that probes 1 and 2 are resistanceless connected to drain. Therefore, we find the longitudinal resistance

$$R_{xx} = \frac{V_1 - V_2}{I_s} = \frac{V_3 - V_4}{I_s} = 0 \tag{7.36}$$

and the Hall resistance

$$R_{xy} = \frac{V_3 - V_1}{I_s} = \frac{V_4 - V_2}{I_s} = \frac{h}{2je^2} \tag{7.37}$$

Within the edge state picture, the quantized Hall resistance is obtained, and the longitudinal resistance vanishes. The accuracy of the quantization is so much more accurate than in a QPC because backscattering is greatly suppressed. Let us now consider what happens as we increase the magnetic field, such that the uppermost occupied LL gets depleted. The corresponding edge state, which is the innermost occupied one, is depopulated as well. Since the velocity in the x-direction of the electrons in edge state j is given by

$$v_j(k_x) = \frac{1}{\hbar}\frac{\partial E_j(k_x)}{\partial k_x} = \frac{1}{\hbar}\frac{\partial V(y(k_x))}{\partial y}\frac{\partial y}{\partial k_x} = \frac{1}{eB}\frac{\partial V(y)}{\partial y} \tag{7.38}$$

where we have used $y(k_x) = \hbar k_x/eB$, the velocity of the electrons approaches zero as the edge state gets depleted. As a consequence, the edge state begins to soften and the electron trajectories penetrate into the bulk. Finally, the electrons can percolate all the way to the opposite edge, backscattering sets in, and the conductance quantization vanishes.

Haug et al. [145] have performed an instructive experiment related to this picture (see Fig. 7.19). A gate stripe extends across a Hall bar inside an area that can be measured by four voltage probes. Biasing the gate tunes the electron density, and thus the number of occupied Landau levels, underneath. If the filling factor under the gate is smaller than outside the gated area, edge states get redirected at the gate. This changes the transmission probabilities in Eq. (7.34).

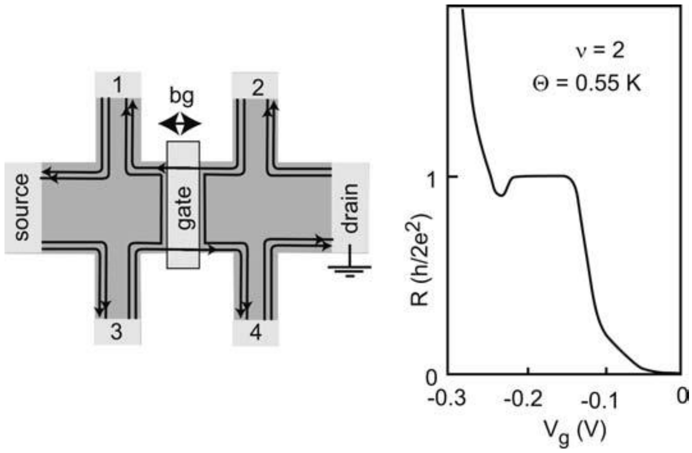

Fig. 7.19 Left: Sample geometry to control backscattering between edge states. A top gate covers the Hall bar in between four voltage probes. At suitable gate voltages, the inner one of the two edge states gets reflected. Right: For a 2DEG in the regime of filling vector 2, with spin-split edge states, a plateau at $R_{xx} = h/2e^2$ is observed as a function of the gate voltage, once the reflection of the inner edge state at the gate is complete. After [145]. The dip around a gate voltage of $-0.2\,\mathrm{V}$ can be explained within a trajectory network formed below the gate.

In Exercise E7.3 the resistances of this system will be calculated. The result for filling factor N in the ungated region and M in the gated region is

$$\begin{aligned}
R_{12} = R_{34} &= \frac{h}{e^2}\left(\frac{1}{M} - \frac{1}{N}\right) \\
R_{13} = R_{24} &= \frac{h}{e^2}\frac{1}{N} \\
R_{14} &= \frac{h}{e^2}\left(\frac{1}{M} - \frac{2}{N}\right) \\
R_{23} &= \frac{h}{e^2}\frac{1}{M}
\end{aligned} \tag{7.39}$$

Note that the results of some measurements now depend on the direction of the magnetic field.

The Landauer–Büttiker formalism is a powerful tool, which allows to treat a variety of problems very elegantly. Further examples are treated in the exercises.

7.3.2
Edge channels

So far, we have interpreted edge states as guiding centers of electron trajectories in strong magnetic fields. Within this picture, the trajectories of electrons moving in different edge states intersect, and we may expect a strong inter-