while the incoming wave functions are related to the outgoing wave functions via $a_3 = b_2 e^{i\theta}$ and $a_2 = b_3 e^{i\theta}$. Let us further assume that a wave is incoming only from the left with amplitude 1, $a_1 = 1$, and no left-moving wave exists to the right-hand side of the double barrier, $a_4 = 0$. This results in a vector $\vec{b}$ of outgoing amplitudes as a function of incoming amplitudes $\vec{a} = (1, b_3 e^{i\theta}, b_2 e^{i\theta}, 0)$, related by

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} r_1 & t_1 & 0 & 0 \\ t_1 & r_1 & 0 & 0 \\ 0 & 0 & r_2 & t_2 \\ 0 & 0 & t_2 & r_2 \end{pmatrix} \begin{pmatrix} 1 \\ b_3 e^{i\theta} \\ b_2 e^{i\theta} \\ 0 \end{pmatrix}$$

Solving for the transmission amplitude b_4 gives

$$b_4 = \frac{t_1 t_2 e^{i\theta}}{1 - r_1 r_2 e^{2i\theta}}$$

leading to the transmission amplitude $T = b_4^* b_4$ of Eq. (8.12). In this particular example, we could easily guess the result by summing up the interference paths. In more complex structures, however, it may not be so easy to do this, and the s-matrices prove to be very useful. We will see an example of this below.

Note that thermal smearing has been neglected. It will be discussed in Exercise E8.4.

Owing to inelastic scattering events, electrons may lose their phase coherence as they traverse the double barrier. In the case of complete incoherence, we do not have to sum up the transmission amplitudes, but rather the transmission probabilities of all trajectories. In that case, the result is

$$T_{\mathrm{sd}}^{\mathrm{inc}} = T_1 T_2 + T_1 R_2 R_1 T_2 + \cdots = \frac{T_1 T_2}{1 - R_1 R_2} \tag{8.14}$$

It should be noted that, in real samples, transport is quite often partly coherent. M. Büttiker found an elegant model for this general situation [46]. The incoherent part of the transmission is modeled by a reservoir in between the barriers, which absorbs and re-ejects those electrons whose phase coherence gets lost.

We conclude this section by discussing the transmission of a quantum ring in terms of the s-matrix formalism. Earlier on, we already studied the transmission of an open ring as a function of the magnetic field, which revealed the Aharonov–Bohm effect. The spectrum of an isolated ring is also well known: in the simplest model, a one-dimensional wire (length $2\pi R$) is bent into a ring, imposing periodic boundary conditions

$$\ell\lambda = 2\pi R, \qquad \ell = 0, \pm 1, \pm 2, \ldots$$

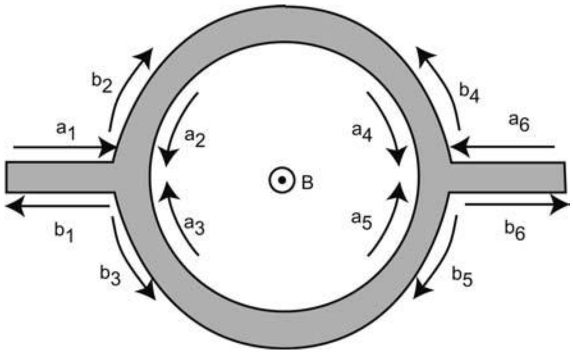

Fig. 8.14 Schematic sketch of the quantum ring under study and the nomenclature of the partial wave functions.

where λ is the electronic wavelength. As a consequence, the wave number is quantized in units of $1/R$. A magnetic field perpendicular to the plane of the ring induces a phase shift of $\Delta\phi = 2\pi\Phi/\Phi_0$, where $\Phi = BA$ is the magnetic flux through the ring (A denotes the ring area), and $\Phi_0 = h/e$ is the magnetic flux quantum. This corresponds to a magnetic wave vector of $k_m = \Delta\phi/2\pi R = (1/R)\Phi/\Phi_0$, and the energy spectrum is given by

$$E_\ell = \frac{\hbar^2}{2m^* R^2}(k_\ell + k_m)^2 = \frac{\hbar^2}{2m^* R^2}(\ell + \Phi/\Phi_0)^2 \tag{8.15}$$

The states are characterized by their angular momentum $\hbar\ell$. This energy spectrum is treated in Exercise E8.3.

Suppose we now couple the ring to two reservoirs to the left and right via tunable tunnel barriers. How will the spectrum of the isolated ring evolve into the Aharonov–Bohm effect observed in open rings? The s-matrices offer a very elegant way to study this evolution. For simplicity, we assume that both tunnel barriers are equal and that the two branches of the ring have the same length (Fig. 8.14).

The junction can be described by the so-called Shapiro matrix

$$(s_{\mathrm{Sh}}) = \begin{pmatrix} c & \sqrt{\epsilon} & \sqrt{\epsilon} \\ \sqrt{\epsilon} & a & b \\ \sqrt{\epsilon} & b & a \end{pmatrix}$$

Here c (a) represent the reflection amplitudes for electrons hitting the junction from lead 1 (2 or 3, respectively), while $\sqrt{\epsilon}$ and b are transmission amplitudes. Unitarity of the s-matrix is given for

$$\epsilon = \tfrac{1}{2}(1 - c^2), \qquad a = -\tfrac{1}{2}(1 + c), \qquad b = \tfrac{1}{2}(1 - c)$$

or

$$\epsilon = \tfrac{1}{2}(1 - c^2), \qquad a = \tfrac{1}{2}(1 - c), \qquad b = -\tfrac{1}{2}(1 + c)$$

The second set of relations corresponds to two ring branches which become decoupled from each other as c approaches zero. Therefore, the first solution describes the situation of interest. Since c is a measure of the coupling of the ring to the leads, we will express the transmission T_{ring} of the ring as a function of c. The matrix (s_{Sh}) is the s-matrix for the left and right junction. The incoming amplitudes are coupled to the outgoing ones via $\vec{b}_{\text{l,r}} = (s_{\text{Sh}})\vec{a}_{\text{l,r}}$ with $\vec{b}_{\text{l}} = (b_1, b_2, b_3)$, $\vec{b}_{\text{r}} = (b_4, b_5, b_6)$, $\vec{a}_{\text{l}} = (a_1, a_2, a_3)$, and $\vec{a}_{\text{r}} = (a_4, a_5, a_6)$. As above, we assume a wave incoming from the left only, with amplitude 1, and denote the phase collected from the vector potential by ϕ, such that the incoming and outgoing waves inside the ring are related via

$$\vec{a} = (\vec{a}_{\text{l}}, \vec{a}_{\text{r}}) = (1, b_4 e^{i\theta} e^{i\phi}, b_5 e^{i\theta} e^{-i\phi}, b_2 e^{i\theta} e^{-i\phi}, b_3 e^{i\theta} e^{i\phi}, 0)$$

This leads to the system of equations

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{pmatrix} = \begin{pmatrix} c & \sqrt{\epsilon} & \sqrt{\epsilon} & 0 & 0 & 0 \\ \sqrt{\epsilon} & a & b & 0 & 0 & 0 \\ \sqrt{\epsilon} & b & a & 0 & 0 & 0 \\ 0 & 0 & 0 & a & b & \sqrt{\epsilon} \\ 0 & 0 & 0 & b & a & \sqrt{\epsilon} \\ 0 & 0 & 0 & \sqrt{\epsilon} & \sqrt{\epsilon} & c \end{pmatrix} \begin{pmatrix} 1 \\ b_4 e^{i\theta} e^{i\phi} \\ b_5 e^{i\theta} e^{-i\phi} \\ b_2 e^{i\theta} e^{-i\phi} \\ b_3 e^{i\theta} e^{i\phi} \\ 0 \end{pmatrix} \tag{8.16}$$

The transmission probability is given by $T_{\text{ring}}(c, \theta, \phi) = b_6^* b_6$. After solving Eq. (8.16) for b_6 and after some algebra, one finds the somewhat lengthy expression

$$T_{\text{ring}}(c, \theta, \phi) = b_6^* b_6 = \frac{16(1-c^2)^2 \cos^2\phi \sin^2\theta}{A + B + C + D + E} \tag{8.17}$$

with

$$\begin{aligned} A &= 5 - 4c + 6c^2 - 4c^3 + 5c^4 \\ B &= (1+c)^4 \cos^2(2\phi) \\ C &= -4(1-c)^2(1+c^2)\cos(2\theta) \\ D &= -2(1+c)^2 \cos^2\phi[2(1+c^2)\cos(2\theta) - (1-c)^2] \\ E &= 8c^2 \cos(4\theta) \end{aligned}$$

Fig. 8.15 shows how the transmission as a function of the dynamic phase θ and the magnetic phase ϕ evolves as the reflection amplitude is reduced. Fig. 8.15(a) corresponds to an open ring, showing essentially Aharonov–Bohm oscillations. Note that here the phase coherence length is infinite. In order to recover the sinusoidal magneto-oscillations typical for the Aharonov–Bohm effect, we would have to expand Eq. (8.17) in a Fourier series and plot

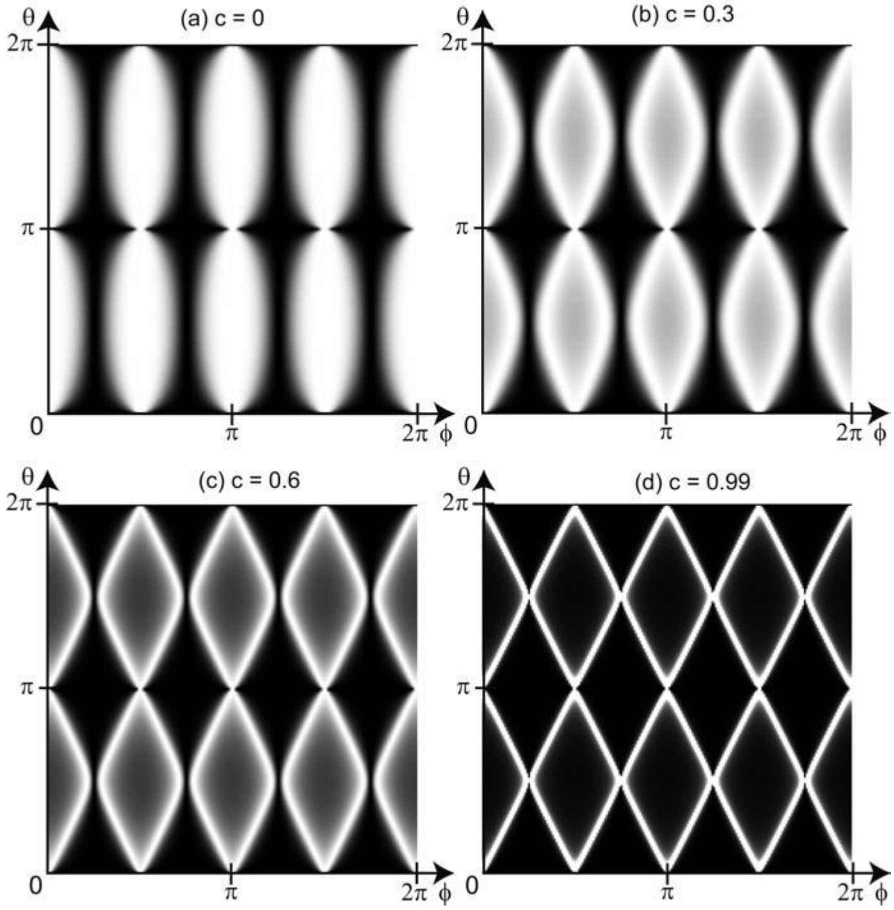

Fig. 8.15 Transmission of an ideal quantum ring as a function of θ and ϕ for different reflection amplitudes at the ring entrances. Black corresponds to $T_{\text{ring}} = 0$, white to $T_{\text{ring}} = 1$.

the first order only. The second order gives the Altshuler–Aronov–Spivak oscillations. Fig. 8.15(d) shows the transmission for a reflection amplitude close to 1 (namely $c = 0.99$). Here, the parabolas of Eq. (8.15) are found (remember that $E \propto \theta^2$). In Figs. 8.15(b) and (c), the transmission is plotted for $c = 0.2$ and 0.4, respectively. Hence, as c increases, the transmission gets more and more concentrated at the edges of the ellipsoidal regions of high transmission in Fig. 8.15(a). Simultaneously, the shape of these ellipsoid-like regions evolves into diamond-like structures.